Preston Corp
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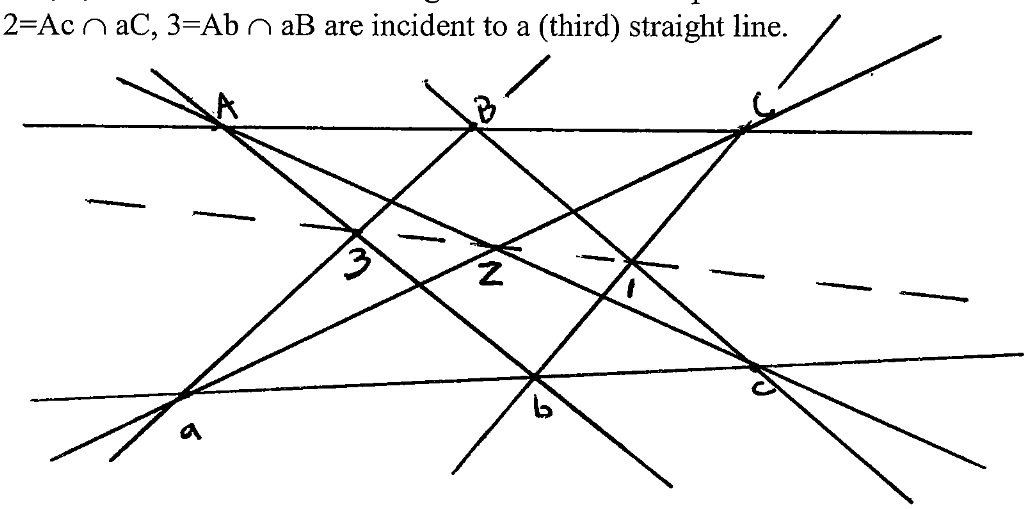
MATH 4513

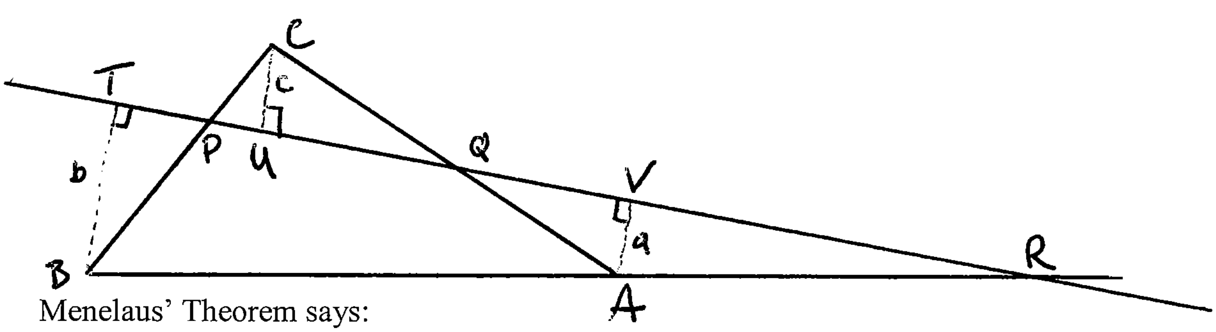
# Pappus' (Hexagon) Theorem

Pappus of Alexandria is considered by some to have been the last of the great Greek geometers. He contributed much to the world of mathematics, and namely to the discipline of geometry. The time period of his life is up for debate, but most would agree that it spanned from around the late third century, to the early to mid-fourth century. Pappus lent many contributions, thoughts, corollaries, and theorems to the world of mathematics. One of which is his Hexagon Theorem. It is this theorem that is the concentration of this paper. It is stated as follows:

Let three points A, B, C be incident to a single straight line and another three points a, b, c incident to another straight line. Then three pair wise intersections $1 = Bc \cap bC$, $2 = Ac \cap aC$, $3 = Ab \cap aB$ are incident to a (third) straight line.




The proof of this theorem can be pretty rigorous, but with the aide of another theorem is fairly simple. This theorem is Menelaus' theorem. This is one in which we were asked to prove on problem four of our midterm. Its proof itself is fairly simple. Let's examine Menelaus' theorem to set the stage for the proof of Pappus' theorem. Consider the diagram:




Menelaus' Theorem says:

Let three points P, Q, and R lie respectively on the sides CB, AC, and AB of ΔABC. Then the points are collinear iff:

$$\frac{BP}{PC}\cdot\frac{CQ}{QA}\cdot\frac{AR}{RB}=1$$

Let us now show this.

From this diagram, we can see many things. First after applying the correct tests, we see three sets of similar triangles:

$\Delta TBP \approx \Delta UCP$ $\Delta UCQ \approx \Delta VAQ$ $\Delta VRA \approx \Delta TRB$

From these similarities, the following ratios follow:

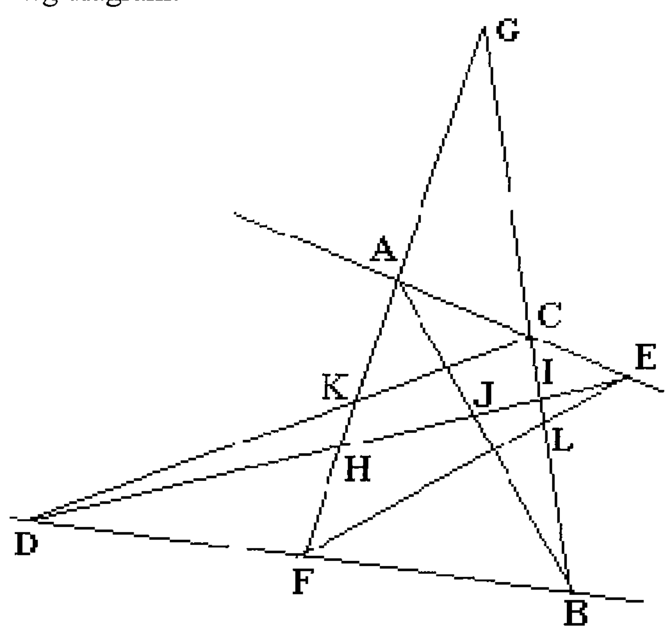
$$\frac{BP}{PC}=\frac{TB}{CU}=\frac{b}{c},\qquad \frac{CQ}{QA}=\frac{CU}{VA}=\frac{c}{a},\qquad \frac{AR}{RB}=\frac{AV}{BT}=\frac{a}{b}$$

Combining these we get the following:

$$\frac{BP}{PC}\cdot\frac{CQ}{QA}\cdot\frac{AR}{RB}=\frac{b}{c}\cdot\frac{c}{a}\cdot\frac{a}{b}=1$$

That concludes the evaluation of Menelaus' theorem, which is the foundation of the proof of Pappus' theorem in which we will now prove.

Consider the following diagram:



Our objective is to prove that points K, J, L are collinear by using the converse of Menelaus' theorem which then will prove Pappus' theorem. To attain this fact we must look at ΔGHI and show:

$$\frac{HK}{GK}\cdot\frac{GL}{IL}\cdot\frac{IJ}{HJ}=1$$

Let us again look at ΔGHI from the diagram, we see five transversals through it: DKC, AJB, ELF, ACE, and DFB. Applying Menelaus' theorem to each of these transversals:

| Transversal | Menelaus' Theorem |
|---|---|
| DKC | $\frac{HK}{GK}\cdot\frac{GC}{IC}\cdot\frac{ID}{HD}=1$ |
| AJB | $\frac{HA}{GA}\cdot\frac{GB}{IB}\cdot\frac{IJ}{HJ}=1$ |
| ELF | $\frac{HF}{GF}\cdot\frac{GL}{IL}\cdot\frac{IE}{HE}=1$ |
| ACE | $\frac{HE}{IE}\cdot\frac{IC}{GC}\cdot\frac{GA}{HA}=1$ |
| DFB | $\frac{HD}{ID}\cdot\frac{IB}{GB}\cdot\frac{GF}{HF}=1$ |

Multiplying these five transversals produces:

$$\left(\frac{HK}{GK}\cdot\frac{GC}{IC}\cdot\frac{ID}{HD}\right)\cdot\left(\frac{HA}{GA}\cdot\frac{GB}{IB}\cdot\frac{IJ}{HJ}\right)\cdot\left(\frac{HF}{GF}\cdot\frac{GL}{IL}\cdot\frac{IE}{HE}\right)\cdot\left(\frac{HE}{IE}\cdot\frac{IC}{GC}\cdot\frac{GA}{HA}\right)\cdot\left(\frac{HD}{ID}\cdot\frac{IB}{GB}\cdot\frac{GF}{HF}\right)=1$$

After canceling terms, we are left with the product of ratios:

$$\frac{HK}{GK}\cdot\frac{GL}{IL}\cdot\frac{IJ}{HJ}=1$$

This product of ratios is the original product that we were after to prove that points K, J, L are collinear using the converse of Menelaus' theorem. Furthermore, proving Pappus' theorem.