MarQuetta Clayton
Cody McGoodwin

Pi is Irrational

Proof by contradiction:
Assume that π is rational, then π^2 must also be rational. Therefore π^2=a/b, where a, b € Z, b ≠ 0. The lim(a→∞) a^n/n!=0, then there exist an M>0 such that if n≥ M, a^n/n! < 1/ π, or πa^n/n! < 1.

Now choose an integer N≥M, then πa^N/N! < 1

Define polynomial:

$$f(x) = x^N (1-x)^N / N!.$$

$$f(x) = \frac{1}{N!}(C_N x^n + C_{N+1} x^{N+1} + C_{N+2} x^{N+2} + \dots + C_{2N} x^{2N})$$

Where C_N is an integer. For integers k N≤ k ≤ 2N, then $f^{(k)}(x) = \frac{1}{N!} \sum_{n=k}^{2N} \frac{n!}{(n-k)!} C_n x^{n-k}$
Notice that $f^{(0)}(x) = (-1)^0 f^0(1-x)$ since
$f(x) = \frac{x^N(1-x)^N}{N!}$ and $f(1-x) = \frac{(1-x)^N x^N}{N!}$.

Suppose its true for an integer k, $f^{(k)}(x) = (-1)^k f^{(k)}(1-x)$.
Then k+1, $f^{(k+1)}(x) = \frac{d}{dx}[f^{(k)}(x)] = \frac{d}{dx}[(-1)^k(-1)f^{(k+1)}(1-x)] = (-1)^{(k+1)} f^{(k+1)}(1-x)$

So the formula is true for all nonnegative integers by induction.

For values of k such that 0≤ k ≤ N each term in the expansion of
$f^{(k)}(0) = 0$ and $f^{(k)}(1) = (-1)^{(k)} f^{(k)}(0) = 0$

For values of k such that N ≤ k ≤ 2N the only term that does not have a positive integer power of x is the first one. This is when n = k. This means $f^{(k)}(0) = \frac{k! C_N}{N!}$, is an integer since, k ≥ N, then $f^{(k)}(1) = (-1)^k f^{(k)}(0)$, is also an integer.

Define another function:

$$F(x) = b^N \sum_{j=0}^{N} (-1)^j \pi^{2N-2j} f^{(2j)}(x) = b^N \sum_{j=0}^{N} (-1)^j \pi^{2(N-j)} f^{2j}(x) = \sum_{j=0}^{N} (-1)^j a^{N-j} b^j f^{(2j)}(x)$$

It now follows that F(0) and F(1) are both integers and thus the F(0) + F(1) is an integer.

Define:
g(x) = F'(x)sin(πx) – πF(x)cos(πx)

It follows that:
F''(x)sin(πx) + πF'(x)cos(πx) – πF'(x)cos(πx) + π^2F(x)sin(πx) = [F''(x) + π^2F(x)]sin(πx).

Now:

$F(x) = b^N[\pi^{2N} f(x) - \pi^{2N-2} f''(\pi) + \pi^{2N-4} f^{(4)}(x) - \dots + (-1)^j f^{2N}(x)$, and so
$F''(x) = b^N[\pi^{2N} f''(x) - \pi^{2N-2} f^{(4)}(x) + \pi^{2N-4} f^{(6)}(x) - \dots + (-1)^j \pi^2 f^{2N+2}(x)]$

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Also:
$\pi^2 F(x) = b^N[\pi^{2N+2} f(x) - \pi^{2N} f''(x) + \pi^{2N-2} f^4(x) + \ldots + (-1)^j \pi^2 f^{(2N)}(x)]$

Notice since f(x) is a polynomial of degree 2N, $f^{(2N+2)}(x) = 0$ for all x.
Thus:
$F''(x) = \pi^2 F(x) = b^N \pi^{(2N+2)} f(x)$

So:
$g'(x) = b^N \pi^{2N+2} f(x) \sin(\pi x) = \frac{a^N}{\pi^{2N}} \pi^{2N+2} f(x) \sin(\pi x) = \pi^2 a^N f(x) \sin(\pi x)$

Since g(x) is continuous on [0,1], and g' exist on (0,1), by the Mean Value Theorem, there exists a c € (0,1) such that g(1) – g(0) = g'(c).

Now:
g(1) = F'(0)sin π – πF(1)(-1) = πF(1) and g(0) = F'(0)sin(0) – πF(0)(1) = - πF(0)

So:
g(1) – g(0) = π F(1) + π F(0) = π [F(1) + F(0)] and thus,
$\pi[F(1) + F(0)] = \pi^2 a^N f(c) \sin(\pi c)$. $F(1) + F(0) = \pi a^N f(c) \sin(\pi c)$

Since:
0< c <1, 0 < sin(πc) ≤ 1 and also 0< 1-c < 1.

Since:
$f(c) = \frac{c^N(1-c)^N}{N!}$, it follows that $0 < \pi a^N f(c) < \frac{\pi a^N}{N!} < 1$

It now follows that $0 < \pi a^N f(c) \sin(\pi c) < 1$ and so 0 < F(1) + F(0) < 1. Since F(1) + F(0) is an integer, there must exist an integer between 0 and 1. Therefore it's a contradiction.
QED.

MarQuetta Clayton
Cody McGoodwin

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· Cody McGoodwin
Marqueta Clayton

2nd Part

$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots$ $\qquad e = \sum_{n=0}^{\infty} \frac{1}{n!}$

assume $e = \frac{p}{q}$ p,q integers

$q!e = q! + \frac{q!}{1!} + \frac{q!}{2!} + \frac{q!}{3!} + \cdots + \frac{q!}{q!} + \cdots$

$q!e$ is an integer since $e = \frac{p}{q}$

so $\quad q!\left(\frac{1}{(q+1)!} + \frac{1}{(q+2)!} \cdots\right) = q!e - \left(q! + \frac{q!}{1} + \cdots + \frac{q!}{q!}\right)$

$\frac{1}{(q+1)} + \frac{1}{(q+1)(q+2)} + \frac{1}{(q+1)(q+2)(q+3)} \cdots < \frac{1}{q+1} + \frac{1}{(q+1)^2}$ where $\frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{(n+1)^2} = 1$

since $(q+1)!$ goes to 0 sooner than $(q+1)^n$ $n \in \mathbb{Z}$
the above is true

e is also transcendental, but the proof is lengthy

now that we know e and pi are irrational and transcendental
we will tackle Hilbert's 7th problem.

$\pi + e$ and πe irrational or transcendental

$\pi + e$, $\pi - e$ adding the two together $(\pi + e) + (\pi - e) = 2\pi$
πe, $\frac{\pi}{e}$ multiply the two we get $\frac{\pi}{e}\pi e = \pi^2$
both transcendental solutions

$\pi + e$, πe however eluded both of us

Cody McGoodwin
Marqueta Clayton

Bibliography for Part II

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