Mathematics in Mainstream Society

Introduction : Mark and Don

Introduction to Mathematic Games:

Today many people play games such as "Sudoku". Mathematical games have been seen throughout history. The egyptians and chinese had many different games or puzzles. One of which I will leave you with is that of the magic square. We have two variations of this game and Mark will show, and I will show the other.(4)

(show 4x 4 puzzle)
(show 3x3 puzzle)

Today we would like to show you some uses for Binary operations in such games:

Nim:

(ask for 2 volunteers) (Play game of nim on board)

Variants of Nim have been played since ancient times. The game is said to have originated in China (it closely resembles the Chinese game of Tsyanshidzi, or "picking stones"), but the origin is uncertain; the earliest European references to Nim are from the beginning of the 16th century. Its current name was coined by Charles L. Bouton of Harvard University, who also developed the complete theory of the game in 1901, but the origins of the name were never fully explained. The name is probably derived from German *nimm!* meaning "take!", or the obsolete English verb *nim* of the same meaning. Some people have noted that turning the word *NIM* upside-down and backwards results in *WIN*, but this is probably just a coincidence. (1)

Explanation:

This solutions was given by C.L. Bouton: (1,3)

There are two versions to this game one being that the last piece taken loses known as a "misére" game, and a normal version where the last piece taken wins. The first player should win with optimal play if the nim sum of thesizes of the heaps is non zero. The key to the game is the binary sum of the heap sizes. This will be written as $x \oplus y$ as to not be mistaken for ordinary sums.

An example of this is as follows:

Binary	Decimal	Heap
011_2	3_{10}	A
100_2	4_{10}	B
101_2	4_{10}	C

010_2	2_{10}	$A \oplus B \oplus C = 2$

The winning strategy is to leave the NimSum = 0

To find out which move to make, let S =the nim sum of all the heap sizes. Then take the nim sums of the heaps:

$$A \oplus X = 3 \oplus 2 = 1$$
$$B \oplus X = 4 \oplus 2 = 6$$
$$C \oplus X = 5 \oplus 2 = 7$$

After doing such the winning strategy is to play in the heap that decreases (A).

The winning move is to reduce A by two items.

Leaving the Nim Sum of the Rows as

A=1
B=4
C=5

The total Nim sum is as follows

$$A \oplus B \oplus C = 1 \oplus 4 \oplus 5 = 0$$

Shown here as
$001+100+101= 000_2$

From here the winning move is to leave an odd number of heaps of size 1.

Proof:

Thm: In NIM the first player has a winning strategy iff the nim sum of the sizes of the heaps is nonzero; otherwise the second player has the winning strategy.

Proof:

Note 1: The nim-sum obeyts the usual associative and commutative laws of addition and also satisfies $x \oplus x = 0$.

Let $x_1,\ldots,x_n$ be the sizes of the heaps before a move, and $y_1,\ldots,y_n$ the corresponding sizes after a move. Let $s = x_1 \oplus \ldots \oplus x_n$ and $t = y_1 \oplus \ldots \oplus y_n$.

If the move was in heap k, we have $x_i = y_i$ for all $i \neq k$ and $x_k \succ y_k$

By the properties above we have:

$$
\begin{aligned}
t &= 0 \oplus t \\
&= s \oplus s \oplus t \\
&= s \oplus (x_1 \oplus \ldots \oplus x_n) \oplus (y_1 \oplus \ldots \oplus y_n) \\
&= s \oplus (x_1 \oplus y_1) \oplus \ldots \oplus (x_n \oplus y_n) \\
&= s \oplus 0 \oplus \ldots \oplus 0 \oplus (x_k \oplus y_k) \oplus 0 \oplus \ldots \oplus 0 \\
&= s \oplus x_k \oplus y_k \, (*)
\end{aligned}
$$

Lemma 1. if s = 0 then $t \neq 0$ no matter what move is made
Lemma 2 if $s \neq 0$ it is possible to make a move so that $t = 0$

Proof of Lemma 1 if there is no possible move then Lemma 1 is true and the first player loses the normal play game by defn. Otherwise, any move In heap k will produce $t = x_k \oplus y_k$ from (*)

Proof of Lemma 2 led d be the leftmost nonzero bit in the binary represtation of s, and choose k s.t. the d^{th} bit of x_k is also nonzero. (this must exist otherwise the d^{th} bit of s would be 0.) then letting $y = s \oplus x_k$ we claim that $y_k < x_k$ all bits to the left of d are the same in x_k and y_k, bit d decreases from 1 to 0 (decreasing the value by 2^k) and any change in the remaining bits will amount to at most 2^k-1. the first player can thus make a move by taking x_k-y_k objects from heap k.

The normal play strategy is for the player to reduce the heaps to size 0 or 1, and the misére strategy is to do the opposite.

There is also a similar game that is yet to have a mathematical solution but is believed to.
Chomp:

Chomp is a game played on a partially ordered set p with smallest element 0.A move consist of picking an elemnt x of P and removing x and all larger elemts from P. whoever picks 0 Loses.

Existance of a Winning Strategy?

If p has a largest element 1, different from 0 then a trivial strategy shows that the first player wins. (if picking 1 does not win, it is because the opponent has the devastating reply. But tin that case the first player wins by starting with a.)However, the argument is non-constructive L the winning move is unknown. (2)

Magic Square:

A magic square is a square array of numbers consisting of the distinct positive integers 1, 2, ..., n^2 arranged such that the sum of the n numbers in any horizontal, vertical, or main diagonal line is always the same number (Kraitchik 1952, p. 142; Andrews 1960, p. 1; Gardner 1961, p. 130; Madachy 1979, p. 84; Benson and Jacobi 1981, p. 3; Ball and Coxeter 1987, p. 193), known as the magic constant.

$$M_2(n) = \frac{1}{n} \sum_{k=1}^{n^2} k = \frac{1}{2} n (n^2 + 1).$$

The first few values are 1, 5, 15, 34, 65, 111, 175, 260, ...

If every number in a magic square is subtracted from $n^2 - 1$, another magic square is obtained called the complementary magic square. A square consisting of consecutive numbers starting with 1 is sometimes known as a "normal" magic square.

8	1	6
3	5	7
4	9	2

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

17	24	1	8	15
23	5	7	14	16
4	6	13	20	22
10	12	19	21	3
11	18	25	2	9

32	29	4	1	24	21
30	31	2	3	22	23
12	9	17	20	28	25
10	11	18	19	26	27
13	16	36	33	5	8
14	15	34	35	6	7

30	39	48	1	10	19	28
38	47	7	9	18	27	29
46	6	8	17	26	35	37
5	14	16	25	34	36	45
13	15	24	33	42	44	4
21	23	32	41	43	3	12
22	31	40	49	2	11	20

64	2	3	61	60	6	7	57
9	55	54	12	13	51	50	16
17	47	46	20	21	43	42	24
40	26	27	37	36	30	31	33
32	34	35	29	28	38	39	25
41	23	22	44	45	19	18	48
49	15	14	52	53	11	10	56
8	58	59	5	4	62	63	1

The unique normal square of order three was known to the ancient Chinese, who called it the Lo Shu. A version of the order-4 magic square with the numbers 15 and 14 in adjacent middle columns in the bottom row is called Dürer's magic square. Magic squares of order 3 through 8 are shown above.

The magic constant for an nth order general magic square starting with an integer A and with entries in an increasing arithmetic series with difference D between terms is

$$M_2(n; A, D) = \frac{1}{2} n [2a + D(n^2 - 1)]$$

(Hunter and Madachy 1975).

It is an unsolved problem to determine the number of magic squares of an arbitrary order, but the number of distinct magic squares (excluding those obtained by rotation and reflection) of order n = 1, 2, ... are 1, 0, 1, 880, 275305224, ... (Sloane's A006052; Madachy 1979, p. 87). The 880 squares of order four were enumerated by Frénicle de Bessy (1693), and are illustrated in Berlekamp *et al.* (1982, pp. 778-783). The number of 5 x 5 magic squares was computed by R. Schroeppel in 1973. The number of 6 x 6 squares is not known, but Pinn and Wieczerkowski (1998) estimated it to be (1.7745^{+}-0.0016)E19 using Monte Carlo simulation and methods from statistical mechanics. Methods for enumerating magic squares are discussed by Berlekamp *et al.* (1982) and on the MathPages website.

A square that fails to be magic only because one or both of the main diagonal sums do not equal the magic constant is called a semimagic square. If all diagonals (including those obtained by wrapping around) of a magic square sum to the magic constant, the square is said to be a panmangic square (also called a diabolic square or pandiagonal square). f replacing each number n by its square n^2 produces another magic square, the square is said to be a bimagic square (or doubly magic square). If a square is magic for n, n^2, and n^3, it is called a trimagic square (or trebly magic square). If all pairs of numbers symmetrically opposite the center sum to $n^2 + 1$, the square is said to be an associative magic square.

PanMagic Square:

If *all* the diagonals--including those obtained by "wrapping around" the edges--of a magic square sum to the same magic constant, the square is said to be a panmagic square (Kraitchik 1942, pp. 143 and 189-191).

1	15	24	8	17
23	7	16	5	14
20	4	13	22	6
12	21	10	19	3
9	18	2	11	25

The Lo Shu is not panmagic, but it is an associative magic square. Order four squares can be panmagic or associative, but not both. Order five squares are the smallest which can be both associative and panmagic, and 16 distinct associative panmagic squares exist.

Sources:

Wikipedia.com (1)

www.win.tue.nl/~aeb/games/chomp.html (2)

C.L. Bouton: Nim, *a game with a complete mathematical theory*, Annals of Mathematics 3 (1901-1902), 35-39 (3)

http://mathworld.wolfram.com./MagicSquare.html