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Transcendental Numbers

What are transcendental numbers?

Transcendental Numbers are numbers that are not the root of any integer polynomial. Basically, a transcendental number can not be written as

$$a_N(x)^N+a_{N-1}(x)^{N-1}+a_{N-2}(x)^{N-2}+\ \dots\ +a_1(x)+a_0=0$$

where x is the transcendental number. This means that it is not an algebraic number of any degree. This means that transcendental numbers must be irrational, because a rational number is an algebraic number of degree one.

We can see that not all irrational numbers are transcendental numbers. Take $(2)^{1/2}$, which is an irrational number. We can use $1(x)^2+0(x)^1+(-2)$. Substituting in $(2)^{1/2}$ we get $2+0+(-2)=0$. This is not a transcendental number.

How can we prove a number is transcendental?

We first must state The Fundamental Principle of Number Theory. This theorem says there are no integers between 0 and 1. This is pretty easy to see.

The typical proof of a transcendental numbers can be done in six steps:

Step 1: **Assume x is algebraic**. In particular, assume there exists a nonzero polynomial $P(z)$ with coefficients such that $P(x)=0$.
Step 2: **Build an integer**. Using the integer coefficients from the proposed polynomial in Step 1, and some additional features of x, produce an integer N. (This step usually involves some technical results from algebra.)
Step 3: **Show that N is not zero**. Prove that the integer N satisfies $0<|N|$. (Typically this step is the most difficult.)
Step 4: **Give an upper bound for N**. Prove that the integer N satisfies $|N|<1$. (This step usually requires some clever insights from analysis to estimate complicated infinite series or exotic integrals.)

Step 5: **Apply the Fundamental Principle of Number Theory**. We have produced an integer $|N|$ between 0 and 1, so we are faced with a mathematical dilemma. If we have not taken any missteps in our previous steps, then the only possible problem is our assumption in Step 1 that x is algebraic.

Step 6: **Big Finish**. With a great sense of accomplishments, we conclude that x is transcendental.

(Taken from Burger & Tubbs, page 10)

We will use these steps to prove that ${}_{n=1}\sum^{\infty} (10^{-n!})$ is a transcendental number.

Proof that ${}_{n=1}\sum^{\infty} (10^{-n!})$ (Liouville's Constant) is a transcendental number

This is the first number that was ever proven to be transcendental.

In order to prove this, we need to know Liouville's Theorem: Let x be an irrational algebraic number of degree d. Then there exists a positive constant depending only on x, c=c(x), such that for every rational number p/q, the inequality

$$c/q^d \leq |x-(p/q)|$$

is satisfied.

We must also understand that $d \geq 2$. If d=1, then we would have an integer times an irrational number plus an integer. An irrational number times an integer would still give us an irrational number. An irrational number plus an integer would still give us an irrational number, so this cannot equal 0.

First assume that ${}_{n=1}\sum^{\infty} (10^{-n!})$ is algebraic of degree d.

$${}_{n=1}\sum^{\infty} (10^{-n!}) = .110001000000000000000001\ldots$$

The amount of zeros between each one increases with every step, and since it goes to infinity, this continues forever. We can see that the decimal expansion of ${}_{n=1}\sum^{\infty} (10^{-n!})$ is non-terminating and non-repeating. ${}_{n=1}\sum^{\infty} (10^{-n!}) \in$ Q. There exists a c > 0 such that $c/q^d \leq |{}_{n=1}\sum^{\infty} (10^{-n!}) - (p/q)|$ for all $p/q \in$ Q.

We can approximate ${}_{n=1}\sum^{\infty} (10^{-n!})$ with $r_n = {}_{n=1}\sum^{N}(10^{-n!})$. Since N is not infinity, r_n's decimal expansion eventually terminates. $r_n \in$ Q. r_n can then be written as p_n/q_n. We can do this by saying $p_n = 10^{N!}\, {}_{n=1}\sum^{N}(10^{-n!})$ and $q_n = 10^{N!}$. By doing a little algebra, we can eliminate the $10^{N!}$ And we are left with ${}_{n=1}\sum^{N}(10^{-n!})$.

Now we can substitute in our x and p/q.

$$\begin{gathered}|x-(p/q)| = \\ {}_{n=1}\sum^{\infty}(10^{-n!}) - {}_{n=1}\sum^{N}(10^{-n!}) = \\ {}_{n=N+1}\sum^{\infty}(10^{-n!}) = \\ 10^{-(N+1)!}+10^{-(N+2)!}+\ldots= \\ (10^{-1})^{(N+1)!}+(10^{-1})^{(N+2)!}+\ldots\end{gathered}$$

If we add terms 10^{-m} where m ϵ Z, m > (N+1)! and m is not factorial, this would include all integer powers of 10^{-1} over (N+1)! This will produce the inequality

$$\begin{gathered}{}_{n=N+1}\sum^{\infty}(10^{-n!}) < {}_{n=N+1}\sum^{\infty}(10^{-n}) \\ {}_{n=N+1}\sum^{\infty}(10^{-n}) = {}_{n=N+1}\sum^{\infty}((1/10)^{-n})\end{gathered}$$

This is a geometric series, which converges to (the first term of the series)/(1-the root of the series), so

$$\begin{gathered}{}_{n=N+1}\sum^{\infty}((1/10)^{-n}) = \\ (10^{-(N+1)!})/(1-(1/10)) = \\ (10^{-(N+1)!})/(9/10) = \\ (10/9)\,(10^{-(N+1)!})\end{gathered}$$

so since

$${}_{n=N+1}\sum^{\infty}(10^{-n!}) < {}_{n=N+1}\sum^{\infty}(10^{-n})$$

then

$${}_{n=N+1}\sum^{\infty}(10^{-n!}) < (10/9)\,(10^{-(N+1)!})$$

and

$$|x-(p/q)| < (10/9)\,(10^{-(N+1)!})$$

Remember Liouville's Theorem states that:

$$c/q^{d} \leq |x-(p/q)|$$

so plugging in our q into the equation, we find

$$c/q^{d} = c/(10^{N!})^{d} = c(10)^{-dN!} < (10/9)\,(10^{-(N+1)!})$$

this leads us to see

$$0 < c(10)^{-dN!} < (10/9)\,(10^{-(N+1)!}) =$$
$$0 < (9/10)c(10)^{-dN!} < (10^{-(N+1)!}) =$$
$$0 < (9/10)c < (10^{-(N+1)!})/\,(10)^{-dN!} =$$
$$0 < (9/10)c < (10)^{dN!-(N+1)!} =$$
$$0 < 9 < (10/c)\,(10)^{dN!-(N+1)!}$$

If $N \geq d$, $d(N!) < (N+1)!$ because

$$d < N =$$
$$d < N+1 =$$
$$d(N!) < (N+1)(N!) =$$
$$d(N!) < (N+1)!$$

Since $d(N!) < (N+1)!$, $dN!-(N+1)! < 0$, so $(10/c)\,(10)^{dN!-(N+1)!} < 1$. This would mean $0 < 9 < 1$. Because of the Fundamental Principle of Number Theory, this cannot happen (and 9 is not less than 1). We have a contradiction if we assume ${}_{n=1}\sum^{\infty} (10^{-n!})$ is an algebraic number, so it must not be. Therefore, ${}_{n=1}\sum^{\infty} (10^{-n!})$ is transcendental.

transcendental number	reference
e	Hermite (1873)
π	Lindemann (1882)
e^{π}	Gelfond
$e^{\pi\sqrt{d}}$, $d \in Z^{*}$	Nesterenko (1999)
$2^{\sqrt{2}}$	Hardy and Wright (1979, p. 162)
$\sin 1$	Hardy and Wright (1979, p. 162)
exponential factorial inverse sum S	J. Sondow, pers. comm., Jan. 10, 2003
$J_0(1)$	Hardy and Wright (1979, p. 162)
$\ln 2$	Hardy and Wright (1979, p. 162)
$\ln 3/\ln 2$	Hardy and Wright (1979, p. 162),
$x_0^{[1]} = 2.4048255\ldots$	Le Lionnais (1983, p. 46)
$\pi + \ln 2 + \sqrt{2}\,\ln 3$	Borwein et al. (1989)
$P_1 = 0.4124540336\ldots$	Dekking (1977), Allouche and Shallit
$P_2 = \sqrt{2} + \ln(1+\sqrt{2})$	
Chaitin's constant Ω_U	
Champernowne constant	
Thue constant	
Liouville's constant L	Liouville (1850)
$\Gamma\left(\frac{1}{3}\right)$	Le Lionnais (1983, p. 46)
$\Gamma\left(\frac{1}{4}\right)$	Chudnovsky (1984, p. 308), Waldschmidt, Nesterenko (1999)
$\Gamma\left(\frac{1}{6}\right)$	Chudnovsky (1984, p. 308)
$\Gamma\left(\frac{1}{4}\right)\pi^{-1/4}$	Davis (1959)
$\zeta(2n)$, $n \in \mathbf{Z} > 1$	

Bibliography

Burger, Edward B. & Tubbs, Robert. Making Transcendence Transparent. Springer Science Business Media, Inc., New York, NY, 2004

Handout from:

http://mathworld.wolfram.com/TranscendentalNumber.html