§12. Quickies and Knights

Quickies. It undoubtedly pays off to do this very important exercise more than twice, on different days, until you can zip right through the questions. You should be able to answer these quickies almost in the time it takes to read the questions.

1. How many ways are there to select 5 women from 16 husband–wife couples?
2. How many ways are there to arrange the 7 letters AAABBBB?
3. How many ways are there to arrange, without having 2 C's together, the 12 letters of AAABBBBCCCCC?
4. How many ways are there to seat 10 people in a row?
5. How many ways are there to seat 10 people at a round table?
6. How many distinguishable dominoes are there, (each of the 2 ends of a domino has 0 to 6 dots carved on it)?
7. How many ways can 14 men and 9 women be seated in a row so that no 2 women sit next to each other?
8. How many ways are there to select 10 cans of soda from 4 different brands?
9. How many ways can 22 cans of beer be handed out to 4 people if everyone must get at least 1 can?
10. How many ways are there to pick 9 cans from among 8 cans of each of 57 varieties?
11. How many ways are there to distribute 5 apples and 8 oranges to 6 children?
12. How many ways are there to select some fruit from 5 apples and 8 oranges, taking at least 1 piece?
13. How many nonnegative integers less than a billion have 5 7's?
14. How many 5-letter words can be formed from the alphabet without repeating any letter?
15. How many ways are there to pair off 8 men with 8 women at a dance?
16. How many positive integer solutions are there to the equation $w + x + y + z = 24$?
17. How many ways are there to pick 12 letters from 12 A's and 12 B's?

18. How many ways are there to pick 18 letters from 12 A's and 12 B's?

19. How many ways are there to pick 25 letters from 12 A's, 12 B's and 12 C's?

20. How many ways are there to select a dozen doughnuts chosen from 7 varieties with the restriction that at least 1 doughnut of each variety must be chosen?

21. How many ways are there to assign 50 agents to 5 different countries so that each country get 10 agents?

22. How many ways are there to put 17 red balls into 12 distinguishable boxes with at least 1 ball in each box?

23. How many ways can 9 dice fall?

24. How many ways can 12 pixies (distinguishable, of course) sit at a round table?

25. How many ways are there to arrange 5 C's and 15 R's such that there are at least 2 R's between any 2 C's?

Consider the following problem.

How many ways are there to select 5 integers from $\{1, 2, \ldots, 20\}$ such that the (positive) difference between any 2 of the 5 is at least 3?

This may be a difficult problem until we think of either choosing or rejecting each integer in turn and associate our selection of the 5 integers with a word consisting of 5 C's (for "Choose") and 15 R's (for "Reject") such that there are at least 2 R's between any 2 C's? Aha! This problem is only the last quickie in disguise. To compute the solution, we start with the diagram ${}_\wedge C_\wedge C_\wedge C_\wedge C_\wedge C_\wedge$, with the 6 wedges indicating places for the insertion of the 15 R's. Each of the inner 4 places must be chosen at least twice, leaving $15 - 8$ arbitrary choices among the 6 places. Thus, we get $\left\langle {6 \atop 7} \right\rangle$ as our answer. We have changed a fairly hard problem into an easy problem. Once seen, such problems now become quickies.

We will use the same technique in a generalization of a classic problem, which is not a quickie and which you might enjoy trying before reading its solution.

The Knights' Quest. How many ways are there to select 4 knights—just the right number for a quest to rescue a fair maiden—from 15 knights sitting at a round table if no adjacent knights can be chosen?

We assume that Lucky Lancelot Pierre is 1 of the knights at the round table. There are 2 cases: either Lancelot goes on the quest or he does not. First, suppose Lancelot goes on the quest. Now, with Lancelot and hence the 2 knights adjacent to Lancelot eliminated from further consideration

16. How many ways are there to put n distinguishable balls into n distinguishable boxes so that exactly 2 boxes are empty?

17. How many ways are there to put 8 indistinguishable balls into 6 distinguishable boxes if the first 2 boxes together have at most 4 balls?

18. How many ways are there to put 8 distinguishable balls into 6 distinguishable boxes if the first 2 boxes together have at most 4 balls?

Four Questions for Thought.

Half of the following questions are easy and half are, at this point, nearly impossible. With experience, we will be able to recognize a very hard problem. What is it that makes half of the following very hard?

- How many ways can we put r distinguishable balls into n distinguishable boxes?
- How many ways can we put r indistinguishable balls into n distinguishable boxes?
- How many ways can we put r distinguishable balls into n indistinguishable boxes?
- How many ways can we put r indistinguishable balls into n indistinguishable boxes?

§15. Three Hour Exams

Practice Exam #1. (Do NOT simplify your answers. You may omit or miss 3 questions without penalty.) **1.** How many 4-letter words are there? **2.** How many 4-letter words have the last letter repeat an earlier letter? **3.** How many ways can 8 individuals sit at a round table? **4.** How many ways can 7 boys and 5 girls line up with all the girls together? **5.** How many ways can 5 1-scoop dishes of ice cream be ordered with repetition if 7 flavors are available? **6.** How many ways can 5 3-scoop dishes of ice cream be ordered with repetition if each scoop can be any 1 of 7 available flavors? **7.** How many ways can 7 oranges and 5 distinguishable toys be distributed to 4 children? **8.** How many ways can 4 A's, 5 B's, and 6 C's be arranged with no B directly following an A? **9.** How many poker hands contain at least 1 card in each suite? **10.** How many ways are there to pick 20 letters from 10 A's, 10 B's, and 10 C's? **11.** How many integer solutions are there to the system $0 < x < y < z < 25$? **12.** How many positive integer solutions are there to the system $x + y + z < 25$? **13.** How many positive integer solutions are there to the system $x+y+z = 25$? **14.** How many ways can we put r indistinguishable balls into s distinguishable boxes? **15.** How many 7 element subsets of the letters of the alphabet have a pair of consecutive letters? **16.** How many arrangements of MISSISSIPPI are there? **17.** How many arrangements of MISSISSIPPI have an S adjacent on each side of the M? **18.** How many arrangements of MISSISSIPPI have both P's precede all the S's? **19.** How many arrangements of MISSISSIPPI have the first S precede the first I and the first I precede the first P? **20.** How many ways can 5 indistinguishable red flags, 7 indistinguishable blue flags, and 11 different, distinct national flags be flown from 14 distinguishable flagpoles?

Practice Exam #2. (Do NOT simplify your answers. You may omit or miss 3 questions without penalty.) **1.** How many ways are there to arrange the letters of MISSISSIPPI? **2.** How many balls must be chosen from 12 red balls, 20 white balls, 7 blue balls, and 8 green balls to be assured that there are 10 balls of the same color? **3.** How many of the 26-letter permutations of the alphabet have no 2 vowels together? **4.** How many 9-element subsets of the letters of the alphabet have no pair of consecutive letters? **5.** How many ways can 6 men and 8 women be seated at a round table with no 2 men next to each other? **6.** How many ways can 9 persons, including Peter and Paul, sit in a row with Peter and Paul not sitting next to each other? **7.** How many 5 letter words are there with no repeated letter if the middle letter is a vowel? **8.** How many arrangements of the letters in MISSISSIPPI have at least 2 adjacent S's? **9.** How many different selections of fruit (including none) can be made from 5 oranges and 7 apples? **10.** How many ways can some (including none and all) of 32 indistinguishable balls be put into 6 distinguishable boxes? **11.** How many ways can we order with repetition of cones allowed 7 double dip ice cream cones from 15 available flavors when the order of the scoops is taken into consideration? **12.** How many ways can we order with repetition of cones allowed 7 double dip ice cream cones from 15 available flavors when the order of the scoops is considered irrelevant? **13.** How many ways are there to distribute 6 apples and 9 oranges to 5 children? **14.** How many ways are there to pick 20 letters from 14 A's and 14 B's? **15.** How many ways are there to put 26 distinguishable flags on 14 distinguishable flagpoles if each flagpole must have at least 1 flag? **16.** How many ways can we put 32 red balls into 12 distinguishable boxes with exactly 3 of the boxes empty? **17.** How many ways are there to distribute 62 indistinguishable white balls and 8 distinguishable numbered balls into 10 distinguishable boxes? **18.** How many ways are there to put 6 indistinguishable red flags, 8 indistinguishable blue flags, and 13 different, distinguishable flags onto 15 distinguishable flagpoles? **19.** How many 10-letter words are there with no 2 adjacent letters the same? **20.** How many arrangements of the word MASSACHUSETTS without any consecutive vowels have an H adjacent to an A?

Practice Exam #3. (Do NOT simplify your answers. You may omit or miss 5 questions without penalty.) **1.** How many ways are there to arrange the letters of MISSISSIPPI? **2.** How many balls must be chosen from 12 red balls, 20 white balls, 7 blue balls, and 8 green balls to be assured that there are 10 balls of the same color? **3.** How many of the 26-letter permutations of the alphabet have no 2 vowels adjacent? **4.** How many ways can 7 distinguishable lions and 4 distinguishable tigers be paraded in line into an arena if no tiger follows directly behind another tiger? **5.** How many ways can 6 men and 8 women be seated at a round table with no 2 men next to each other? **6.** How many ways can 9 persons, including Peter and Paul, sit in a row with Peter and Paul not sitting next to each other? **7.** How many 5-letter words are there in which every 3-letter subword consists of 3 distinguishable letters? **8.** How many arrangements of MISSISSIPPI have the first I precede the first S? **9.** How many different selections of fruit (including none) can be made from 5 oranges and 7 apples? **10.** How many ways can 10 passengers sit in a 10-seat train compartment with 5 seats facing forward and 5 seats facing backward, accommodating 4 who want to face forward and 3 who want to face backward? **11.** How many ways can we order with repetition of cones allowed 7 double dip ice cream cones from 9 available flavors when the order of the scoops is not taken into consideration? **12.** How many ways can 6 speakers be ordered if speaker A must not precede speaker B? **13.** How many ways are there to form 6-element subsets of $\{1, 2, \ldots, 25\}$ such that the largest element is 20? **14.** How many ways are there to form 6-element subsets of $\{1, 2, \ldots, 25\}$ such that the largest element is greater than 20? **15.** How many ways are there to form 6-element subsets of $\{1, 2, \ldots, 25\}$ having no 2 consecutive integers? **16.** How many ways are there to arrange 4 A's, 5 B's, 6 C's, and 7 D's with no consecutive B's? **17.** How many ways are there to arrange 4 A's, 5 B's, 6 C's, and 7 D's with no substring AB occurring? **18.** How many ways are there to seat 7 men and 7 women in a row with men and women alternating? **19.** How many ways are there to seat 7 men and 7 women at a round table with men and women alternating? **20.** How many ways are there to seat 7 inseparable couples at a round table? **21.** How many ways are there to walk a total of 7 blocks north and 4 blocks west while stopping at the shop that is 3 blocks north and 2 blocks west? **22.** How many ways are there to put m distinguishable flags on n indistinguishable flagpoles so that each flagpole has at least 1 flag? **23.** How many ways can 8 indistinguishable red flags, 9 indistinguishable blue flags, and 10 different, distinct national flags be flown from 11 distinguishable flagpoles? **24.** How many ways can 8 indistinguishable red flags, 9 indistinguishable blue flags, and 10 different, distinct national flags be flown from 11 distinguishable flagpoles if each flagpole has at least 1 flag? **25.** How many ways can 8 indistinguishable red flags, 9 indistinguishable blue flags, and 10 different, distinct national flags be flown from 11 distinguishable flagpoles if each flagpole has at least 2 flags?