Solutions to some exercises on outer measure

#21, p. 352. We’ll show that $\mu^*(E) = 1$ for every nonempty subset $E$ of $X$. To see this, observe that the only collection of sets in $S$ which covers $E$ (ignoring the empty set, which has no effect), is the collection consisting of the single set $X$. So, by definition the outer measure of $E$ equals $\mu(X)$, which is 1.

It is easy to verify that the only collection of measurable sets that both $\emptyset$ and $X$ are measurable. (Alternatively, we could use the fact that if $\mu$ is a premeasure on an algebra $A$, then every set in $A$ is $\mu^*$ measurable; this was proved as part of the proof of the Carathéodory-Hahn extension theorem. In this case $S$ is a $\sigma$-algebra and therefore also an algebra, and $\mu$ is clearly a measure on $S$, so it is a premeasure on $S$.)

If $E$ is any subset of $X$ which is neither empty, nor equal to all of $X$, then $E$ is not measurable. To see this, let $x_1$ be an element in $E$ and let $x_2$ be an element in $\mathbb{E}^c$, and let $A = \{x_1, x_2\}$. Then $A, A \cup E$, and $A \sim E$ are all non-empty, so $\mu^*(A) = \mu^*(A \cup E) = \mu^*(A \sim E) = 1$, so $\mu^*(A) \neq \mu^*(A \cup E) + \mu^*(A \sim E)$. Hence $E$ is not measurable.

This proves that the collection of measurable sets in $X$ is exactly the collection $\{\emptyset, X\}$.

#25, p. 357. The only finite disjoint collections of sets $\{E_k\}_{k=1}^n$ in $S$ are the collection $\{\emptyset\}$ and the collection $\{A\}$, with only one set each, so it’s trivial that $\mu$ is finitely additive. To verify that $\mu$ is countably monotone, it’s enough to check that $\mu(A) \leq \mu(X)$, since the only way to cover one set in $S$ by a union of other sets in $S$ is to take $A \subseteq A$. Define a set function $\nu$ on $\Sigma$ by setting $\nu(\emptyset) = 0$, $\nu(A) = 1$, $\nu(X \sim A) = 1$, and $\nu(X) = 2$. Then $\Sigma$ is a $\sigma$-algebra which contains $S$, $\nu$ is a measure on $\Sigma$, and $\nu$ agrees with $\mu$ on $S$.

Notice, however, that $\nu$ does not agree with the outer measure $\mu^*$ induced by $\mu$ on $\Sigma$. In fact, from the definition of $\mu^*$, we see easily that $\mu^*(E) = 2$ whenever $E$ is a subset containing points which are not in $A$, $\mu^*(E) = 1$ whenever $E$ is a nonempty subset of $A$, and $\mu^*(E) = 0$ when $E$ is empty. In particular, $\mu^*(X \sim A) = 2$, so $\mu^*(X \sim A) \neq \nu(X \sim A)$.

We claim that the only $\mu^*$-measurable sets are $\emptyset$ and $X$. To see this, suppose $E$ is any set which is neither empty nor equal to all of $X$. There are two cases to consider: either $E$ contains points which are not in $A$, or $E$ is a subset of $A$. In the first case, $\mu^*(X \cap E) = \mu^*(E) = 2$ and, since $X \sim E$ is not empty, $\mu^*(X - E) \geq 1$. So

$$\mu^*(X \cap E) + \mu^*(X \sim E) = 3 \neq \mu^*(X),$$

which proves that $E$ is not $\mu^*$-measurable. In the second case, when $E \subseteq A$, we have $\mu^*(X \sim E) = 2$ and $\mu^*(X \cap E) = 1$, so again

$$\mu^*(X \cap E) + \mu^*(X \sim E) = 3 \neq \mu^*(X),$$

and again $E$ is not $\mu^*$-measurable.

#26, p. 357. The only disjoint collections of non-empty sets in $S$ whose unions are also in $S$ are collections with just one set, so $\mu$ is trivially finitely additive on $S$. (Note that $[0, 1]$ and $[2, 3]$ are disjoint sets in $S$, but their union is not in $S$.) The only ways to cover one set in $S$ by a union of nonempty sets in $S$ are: $[0, 1] \subset [0, 3]$, $[2, 3] \subset [0, 3]$, and covers in which a set is covered by itself. Since $\mu([0, 1]) \leq \mu([0, 3])$ and $\mu([2, 3]) \leq \mu([2, 3])$, it follows that $\mu$ is countably monotone on $S$. Hence $\mu$ is a premeasure on $S$.

However, unlike the measure in problem 25 above, this premeasure $\mu$ cannot be extended to a measure $\nu$ on any $\sigma$-algebra containing $S$. For any $\sigma$-algebra containing $S$ must also contain the set $[0, 1] \cup [2, 3]$, and since $[0, 1] \cup [2, 3] \subseteq [0, 3]$, if $\nu$ is a measure defined for these sets then we must have

$$\nu([0, 1]) + \nu([2, 3]) = \nu([0, 1] \cup [2, 3]) \leq \nu([0, 3]).$$

Therefore $\nu$ could not be an extension of $\mu$, because

$$\mu([0, 1]) + \mu([2, 3]) = 2 > \mu([0, 3]).$$
It’s not hard to see from the definition of \( \mu^* \) that \( \mu^*(E) = 1 \) for any nonempty subset \( E \) of \( \mathbb{R} \) which is contained in \([0, 3], \) and (using the convention mentioned in the footnote at the bottom of page 350) \( \mu^*(E) = \infty \) for any subset \( E \) of \( \mathbb{R} \) which has at least one point in common with \( \mathbb{R} \sim [0, 3]. \)

The \( \mu^* \)-measurable sets consist of all subsets of the form \( A \cup B, \) where \( A \subseteq \mathbb{R} \sim [0, 3] \) and either \( B = \emptyset \) or \( B = [0, 3]. \) To see this, first let’s check that any such set is measurable.

Suppose \( A \) is any subset of \( \mathbb{R} \sim [0, 3], \) and \( C \) is any subset of \( \mathbb{R}. \) There are three possibilities: either \( C \) is empty, or \( C \subset [0, 3], \) or \( C \) has at least one point in common with \( \mathbb{R} \sim [0, 3]. \) In the first case, clearly
\[
\mu^*(C) = \mu^*(C \cap A) + \mu^*(C \sim A)
\]
is true, because all three measures are zero. In the second case, we have \( \mu^*(C) = 1, \) \( \mu^*(C \cap A) = 0, \) and \( \mu^*(C \sim A) = 1, \) so the above equation is still true. In the third case, we have \( \mu^*(C) = \infty, \) and since the sets \( C \cap A \) and \( C \sim A \) together cover \( C, \) at least one of them must contain a point in \( \mathbb{R} \sim [0, 3], \) so one of the measures on the right side of the equation must be infinite. Therefore the equation again holds in this case. So \( A \) is measurable.

Now \( B = \emptyset \) is measurable, and \( B = [0, 3] \) is measurable because for every nonempty subset \( C \) of \( \mathbb{R}, \) if \( C \subseteq [0, 3] \) then
\[
1 = \mu^*(C) = \mu^*(C \cap B) + \mu^*(C \sim B) = 1 + 0,
\]
and if \( C \) has at least one point in common with \([0, 3]\) then the equation still holds, by the same argument as in the preceding paragraph.

Since all sets \( A \) and \( B \) of the above form are measurable, it follows that the unions \( A \cup B \) of sets of the above form are also measurable.

Now let’s show that any set \( E \) not of the form \( A \cup B \) with \( A \subseteq \mathbb{R} \sim [0, 3] \) and \( B = \emptyset \) or \( B = [0, 3] \) is not measurable. If \( E \) is not of this form, then \( E \) has at least one element \( x_1 \) in common with \([0, 3], \) and there is at least one element \( x_2 \) of \([0, 3]\) which is not in \( E. \) Let \( C = \{x_1, x_2\}. \) Then \( \mu^*(C) = 1, \) \( \mu^*(C \cap E) = \mu^*(\{x_1\}) = 1, \) and \( \mu^*(C \sim E) = \mu^*(\{x_2\}) = 1. \) Hence
\[
1 = \mu^*(C) = \mu^*(C \cap E) + \mu^*(C \sim E) = 2,
\]
so \( E \) is not measurable.

\#27, p. 357. Every outer measure is countably monotone, so if \( \mu^* \) is an extension of \( \mu, \) then because \( \mu^* \) is countably monotone, then \( \mu \) must be also.

Conversely, suppose \( \mu \) is countably monotone. We want to prove that \( \mu^* \) is an extension of \( \mu, \) or in other words that for every set \( E \) in \( \mathcal{S} \), \( \mu^*(E) = \mu(E). \) From the definition of \( \mu^* \) it is clear that \( \mu^*(E) \leq \mu(E), \) so it suffices to prove the reverse inequality. For every \( \epsilon > 0, \) by definition of \( \mu^* \) there exist sets \( E_k \) in \( \mathcal{S} \) such that \( E \subseteq \bigcup_{k=1}^\infty E_k \) and
\[
\sum_{k=1}^\infty \mu(E_k) \leq \mu^*(E) + \epsilon.
\]
Since \( \mu \) is countably monotone, we have
\[
\mu(E) \leq \sum_{k=1}^\infty \mu(E_k).
\]
Combining with the preceding inequality we get
\[
\mu(E) \leq \mu^*(E) + \epsilon,
\]
and since \( \epsilon \) is arbitrary, it follows that \( \mu(E) \leq \mu^*(E), \) as desired.