Notes — The Quest to Calculate Pi

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Introduction

What is \( \pi \)?

What do we know about \( \pi \)?

Why \( \pi \)? How do you calculate \( \pi \)?

Old School \( \pi \)

Babylonian method

Egyptian method

Archimedes (250 BC – 17th c.)

New School \( \pi \)

Arctan method (17th c. – 1980s)

Computers (1980 – )

Conclusion

What don’t we know about \( \pi \)?

Intro

Today John Paul and I will be talking about \( \pi \). It is a journey which has spanned continents and millennia.

I’m sure our title has already riveted your attention to our every word and I’m sure a plethora of questions are racing through your head.

What is \( \pi \)?

Have you asked yourself —

\[
4 \arctan 1 = \pi \quad \frac{\pi}{4} = \frac{1}{1} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \ldots
\]

we use it all the time in math: \( \pi r^2 = A \); \( \pi \cdot \pi = C \);

\[ e^{\pi i} + 1 = 0 \]

3.1415926535 897 9323846 2643383279 5028841971

it shows up in normalization of normal distribution, on the distribution of primes, all over the place.

A quick answer: all geometric definitions equivalent to \( \pi = C \) — known first knowledge of \( \pi \); \( \pi \) = \( C \), but many other places it shows up.

It’s kind of a mystery which is one of the reasons \( \pi \) is so cool.

What exactly do we know about \( \pi \)?

\( \pi \), ratio of the circumference of a circle to its diameter is

irrational (Johann Heinrich Lambert 1761)

transcendental (Ferdinand von Lindemann 1882)

can’t construct, square the circle.
we know
- 1st 0 in decimal expansion - pos. 32
- 1st time all 10 digits occur in a 10 digit block - pos. 60
   in order: 0123456789 - pos. 17,387,594,880
- self referential positions: 1, 16, 470, 44, 899, 79, 873, 884
- Feynman points: six 9s position 768
- decimal digits past 1 trillion 242 million computer programme to find birthdays, messages, addresses etc.

but still much we do not know might also ask...

Why π? - why keep searching for ways to find more digits
- 10 or fewer digits enough significant figures for all practical calculations
- 39 - sufficient to calculate volume of universe (π unleashed)

it possesses people,
Dutch mathematician Ludolph van Ceulen so proud of the 35 digits
of π he calculated, he had it put on his tombstone

why?
- test computer systems, find bugs
- classical and ultimate test bed for numeric analysis
- unsolved questions raised by π in many fields makes it more interesting
- set world records

How do you calculate π?
- today we'll try to give you a snapshot of how it has been done over time

My colleague John Paul Cook will now cover early attempts to calculate π - old school π if you will.

JPC

<table>
<thead>
<tr>
<th>Babylonians</th>
<th>Egyptians</th>
<th>Archimedes</th>
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</thead>
<tbody>
<tr>
<td></td>
<td>3 1/8 = 3.125</td>
<td></td>
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<tr>
<td></td>
<td>4 (8/9)^2 = 3.1604</td>
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<td>4 (13/11)^2 = 3.088</td>
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<td>3</td>
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<td>[ \sqrt{2} + \sqrt{3} = 3.14626 ]</td>
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<td></td>
<td>3 10/71 &lt; π &lt; 3 1/71</td>
<td>3.1408 &lt; π &lt; 3.14285</td>
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</tbody>
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New School \(\pi\)

- Archimedean method used to calculate \(\pi\), expand \(\pi\) to \(2000\) years

- then in second half of 17th century, infinitesimal analysis developed
  - infinitesimal analysis — archaic term for calculus
  - remember 3rd calculus class — series, go back
  - formulations of infinite expressions for \(\pi\) using series based on the arctan function

\[
\arctan x \text{ inverse function of } \tan x \text{ on } (-\pi/2, \pi/2)
\]

on unit circle

\[
\sin x = \frac{y}{r}, \quad \cos x = \frac{x}{r}
\]

the point \(\pi/4\) on unit circle

angle \(\pi/4\) in triangle \(\Delta\) interesting for our purposes

\[
\tan \frac{\pi}{4} = \frac{\frac{\pi}{4}}{\pi/4} = 1 \quad \therefore \quad \frac{\pi}{4} = \tan 1
\]

so, going back to arctan formula \(\arctan 1 = \frac{\pi}{4} \Rightarrow \pi = 4 \arctan 1\)

we can calculate this thanks to the infinite series for arctan

- discovered by James Gregory 1671

Gregory series

\[
\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots
\]

substituting 1, we get the Leibnitz series 1674

\[
\arctan 1 = \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1}
\]

this doesn't converge quickly; 2 billion terms — 9 accurate decimal

quick convergence = finding digits of \(\pi\) with less work; places of \(\pi\)

so, attempts were made to assemble angle \(\pi/4\) from relatively shorter arcs

Simplest: Euler 1738

\[
\frac{\pi}{4} = \arctan \frac{1}{2} + \arctan \frac{1}{3}
\]

\[
\tan (\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}
\]

\[
\frac{1}{2} + \frac{1}{3} = \frac{5}{6}, \quad \frac{1}{2} = \frac{1}{3}, \quad \arctan \frac{1}{2} \approx \arctan \frac{1}{3}
\]

substitute

\[
\tan \alpha = \frac{1}{2}, \quad \arctan \frac{1}{2} = \alpha
\]

\[
\tan \beta = \frac{1}{3}, \quad \arctan \frac{1}{3} = \beta
\]

\[
\frac{\pi}{4} = \arctan \frac{1}{2} + \arctan \frac{1}{3}
\]
\[
\arctan \frac{1}{2} + \arctan \frac{1}{3} = \frac{1}{2} + \frac{1}{3} - \frac{1}{2 \cdot 3.3} - \frac{1}{3 \cdot 3.3} + \frac{1}{2 \cdot 5.5} + \frac{1}{3 \cdot 5.5} - \frac{1}{2 \cdot 7} - \frac{1}{3 \cdot 7} + \ldots
\]

Machin formula:
\[
tan 2\alpha = \frac{2 \tan \alpha}{1 - \tan^2 \alpha}
\]

\[
tan (\alpha - \beta) = \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta}
\]

\[
\text{use } \alpha = \arctan \frac{1}{3} = \frac{1}{3} - \frac{1}{3 \cdot 3.3} + \frac{1}{5 \cdot 3.3} \ldots - \frac{1}{3 \cdot 7} \ldots \text{ easy to calculate}
\]

\[
tan 2\alpha = \frac{5}{12}
\]

\[
tan 4\alpha = \frac{120}{119} \Rightarrow 4\alpha = 4 \arctan \frac{1}{3}
\]

\[
\text{4}\alpha \text{ only slightly larger than } \frac{\pi}{4}; \text{ can find angle } \beta \text{ s.t. } \frac{\pi}{4} = \beta
\]

\[
\frac{\tan 4\alpha - \tan \frac{\pi}{4}}{1 + \tan 4\alpha \cdot \tan \frac{\pi}{4}} = \frac{\frac{120}{119} - 1}{1 + \frac{120}{119}} = \frac{\frac{119}{119} - \frac{119}{119}}{1 + \frac{120}{119}} = \frac{259 - 119}{239} = \frac{259}{119}
\]

\[
\frac{\pi}{4} = 4\alpha - \beta = 4 \arctan \frac{1}{3} - \arctan \frac{1}{239}
\]

\[
\frac{\pi}{4} = 4 \left[ \frac{1}{3} - \frac{1}{3 \cdot 3.3} + \frac{1}{5 \cdot 3.3} \ldots \right] - \left[ \frac{1}{239} - \frac{1}{3 \cdot 239} \ldots \right]
\]

wanted to give idea of convergence but calculation difficult

Others:

Størmer 1896

\[
\frac{\pi}{4} = 6 \arctan \frac{1}{8} + 2 \arctan \frac{1}{5} + \arctan \frac{1}{239}
\]

Gauss used in 1973

\[
\frac{\pi}{4} = 12 \arctan \frac{1}{18} + 8 \arctan \frac{1}{57} - 5 \arctan \frac{1}{239}
\]

These formulas used until 20-25 years ago

1940s advent of computing made more calculations possible

1980s FFT

Computing power increasing all the while

Other formulas - specific to computing

Ramanujan

\[
\frac{\pi}{16} = \frac{\sqrt{8}}{9801} \sum_{n=0}^{\infty} \frac{(4n)!}{n!^4} \frac{252}{(2n)^3 + (-1)^n}
\]

Chudnovsky 1987

See other paper
Gauss AGM
number of \( \pi \) digits doubles each iteration stage
converges quadratically

Borwein Quadratic

all these decimal
hexadecimal expansion is

BBP
do two or three digits

Conclusion JP Cook

3. 14159265
3. 243F6A8B7C