MIDTERM EXAM

MATH 5403

1a) $\delta J[h] = \dfrac{d}{d\varepsilon} J[y+\varepsilon h]\Big|_{\varepsilon=0} = \dfrac{d}{d\varepsilon}\Big|_{\varepsilon=0} \displaystyle\int_0^1 [(y+\varepsilon h)']^3 - (y+\varepsilon h)^2 \, dx$ (2)

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$$= \int_0^1 \{3\,(y'+\varepsilon h')^2 \cdot h' - 2(y+\varepsilon h)\cdot h\}\, dx \Big|_{\varepsilon=0}$$ (2) (2)

$$= \int_0^1 \{3(y')^2 h' - 2yh\}\, dx \;;$$ (3) and

$$\delta^2 J[h] = \frac{d^2}{d\varepsilon^2} J[y+\varepsilon h]\Big|_{\varepsilon=0}$$ (2)

$$= \frac{d}{d\varepsilon} \int_0^1 \{3(y'+\varepsilon h')^2 \cdot h' - 2(y+\varepsilon h)\cdot h\}\, dx \Big|_{\varepsilon=0}$$ (2)

$$= \int_0^1 \{6\,(y'+\varepsilon h')\cdot h' \cdot h' - 2h\cdot h\}\, dx \Big|_{\varepsilon=0}$$ (2)

$$= \int_0^1 \{6y'(h')^2 - \boxed{2h^2}\}\, dx$$ (2)

(1b) If $\hat{y}$ is constant, $\hat{y} \equiv C$, then $\hat{y}' \equiv 0$, so

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$$\delta^2 J[h] = \int_0^1 -2h^2 \, dx .$$ So if $h \in C^1[0,1]$

and $h(0)=h(1)=0$, and h is not identically zero, then $\delta^2 J[h] < 0$. (4) Since a necessary condition for a relative minimum is that $\delta^2 J[h] \geq 0$ (4) for all such h, then $\hat{y}$ cannot be a relative minimum

[25]

2) $F(x,y,y') = y(y')^2$ does not depend on x;

so the Euler-Lagrange equation has the integral

$$C = F - y' F_{y'} = y(y')^2 - y'\,y\,2y' \underset{5}{=} -y(y')^2,$$

so $(y')^2 = \frac{-C}{y} = \frac{D}{y}$, or $y' = \frac{\sqrt{D}}{\sqrt{y}}$, or $\sqrt{y}\,y' = E$,

or $\int \sqrt{y}\,dy = E\int dx$. 5 So $\frac{2}{3}y^{3/2} = Ex + F$, or

$y^{3/2} = Gx + H$, $y = (Gx+H)^{2/3}$. 5 Since $y(0) = 1$,

Then $1 = H^{2/3}$, so $H = \pm 1$.

If $H = 1$ Then $y = (Gx+1)^{2/3}$, so $y(1) = 4^{1/3} \Rightarrow$

$(G+1)^{2/3} = 4^{1/3} \Rightarrow (G+1)^2 = 4 \Rightarrow G = 1$ or $G = -3$. 10

Thus $y = (-3x+1)^{2/3}$ or $y = (x+1)^{2/3}$.

If $H = -1$ Then $y = (Gx-1)^{2/3}$, so $y(1) = 4^{1/3} \Rightarrow$

$(G-1)^{2/3} = 4^{1/3} \Rightarrow (G-1)^2 = 4 \Rightarrow G = 3$ or $G = -1$.

Then $y = (3x-1)^{2/3}$ or $y = (-x-1)^{2/3}$; but these are

the same as $y = (-3x+1)^{2/3}$ and $y = (x+1)^{2/3}$.

3) Here $F(x,y,y') = y^2 - (y')^2$ is independent of x,

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so the Euler-Lagrange equation has the integral

$$C = F - y' F_{y'} = y^2 - (y')^2 - y'(-2y') = y^2 + (y')^2.$$ 5

So $y' = \pm\sqrt{C - y^2}$, or $\int \frac{dy}{\sqrt{C-y^2}} = \pm \int dx$, so 2

(for $y = \sqrt{C} \sin u$) $\int \frac{\sqrt{C} \cos u \, du}{\sqrt{C}\sqrt{1-\sin^2 u}} = \pm x + D$, or

$\int du = \pm x + D$, or $u = \pm x + D$, so

$y = \sqrt{C} \sin(\pm x + D) = E \sin(x + D)$. 5

Since $y(0) = 0$, then $0 = E \sin D$, so we can take $D = 0$ (taking $D = n\pi$ doesn't change the answer).

Therefore $y = E \sin x$. 3

To be a minimizer over the set of all y such that $y(0)=0$, the extremal has to satisfy the free boundary condition $F_{y'} = 0$ at $x = \frac{\pi}{2}$. But $F_{y'} = -2y'$, 5

so this means $y' = 0$ at $x = \frac{\pi}{2}$. Since $y' = E \cos x$, then $y' = 0$ at $x = \frac{\pi}{2}$ for <u>every</u> choice of E. So all functions $y = E \sin x$ are extremals.

4) Here $F(x, y, y', y'') = 360x^2 y - (y'')^2$, and

[20] the Euler-Lagrange equation is

$$F_y - \frac{d}{dx}(F_{y'}) + \frac{d^2}{dx^2}(F_{y''}) = 0 \quad (5) \quad ,$$

or $\quad 360x^2 - 0 + \frac{d^2}{dx^2}(-2y'') = 0,$ (5)

or $\quad y'''' = 180x^2.$ So $\quad y''' = 60x^3 + C,$

$$y'' = 15x^4 + Cx + D,$$

$$y' = 3x^5 + \frac{C}{2}x^2 + Dx + E, \quad (10)$$

$$y = \frac{1}{2}x^6 + \frac{C}{6}x^3 + \frac{D}{2}x^2 + Ex + F,$$

or $\quad y = \frac{1}{2}x^6 + \alpha x^3 + \beta x^2 + \gamma x + \delta$; with $\alpha, \beta, \gamma, \delta$ arbitrary.

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5) Extremals for the problem of varying J while

[10] holding K constant satisfy the equation

$$F_y - \frac{d}{dx}(F_{y'}) = \lambda \left[G_y - \frac{d}{dx}(G_{y'}) \right] \quad (5)$$

for some $\lambda \in \mathbb{R}$. We were given that the extremals were not extremals for J alone or for K alone, so

$F_y - \frac{d}{dx}(F_{y'}) \neq 0$, so $\lambda \neq 0$. Therefore we can write

$$G_y - \frac{d}{dx}(G_{y'}) = \frac{1}{\lambda}\left[F_y - \frac{d}{dx}(F_{y'}) \right]. \quad (5)$$

It follows that y also satisfies the condition for being an extremal for the problem of minimizing K while holding J constant.