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Math 4513
Midterm exam

Name: Key

(You may omit or miss one question without penalty.)

[10] **1.** How many ways can 5 men and 7 women be seated in a row with no 2 men next to each other?

5 Seat the women first: $7!$ ways

$\wedge W \wedge W \wedge W \wedge W \wedge W \wedge W \wedge W \wedge$

Pick 5 of the 8 places to put the men: $\binom{8}{5}$ ways 5

Arrange the men: $5!$ ways

(Note: See below for solution if men are seated first)

Ans. $\boxed{7!\binom{8}{5}5!}$

[10] **2.** How many ways can 13 red balls be put into 5 distinguishable boxes with no box being empty?

First put one ball into each box: 1 way 5

Now distribute the remaining 8 balls among the 5 boxes:

$\boxed{\left\langle {5 \atop 8} \right\rangle \text{ ways}}$ 5

[10] **3.** How many different selections of fruit (including none) can be made from 5 oranges and 8 apples?

4 There are 6 possible choices for the number of oranges, ranging from 0 to 6. For each of these choices, there are 9 possible choices for the number of apples. So the answer is $\boxed{6 \cdot 9}$. 2

4

[10] **4.** How many ways are there to distribute 20 apples and 12 (distinguishable) toys to 10 children?

Number of ways to distribute apples: $\left\langle {10 \atop 20} \right\rangle$ 4

Number of ways to distribute toys: 10^{12} 4

Answer: $\boxed{\left\langle {10 \atop 20} \right\rangle \cdot 10^{12}}$ 2

Alternate solution to #1: Seat the 5 men first. They will have to be separated (5! ways) by 4 of the women: $\wedge M W \wedge M W \wedge M W \wedge M W \wedge M \wedge$. The remaining 3 women can be placed in the 6 indicated spaces in $\left\langle {6 \atop 3} \right\rangle$ ways, then all women arranged in $7!$ ways: Ans = $\boxed{5!\left\langle {6 \atop 3} \right\rangle 7!}$

[10] **5.** How many 10-letter words are there with no 2 adjacent letters the same?

There are 26 choices for the first letter, then 25 choices (5) for each remaining letter (each remaining letter can be any letter except the one preceding it). So the answer is

$$\boxed{26 \cdot (25)^9}$$ (5)

Note: this problem could also be done using P.I.E. See bottom of page ↓

[10] **6.** How many arrangements of NORMANOKLAHOMA have both M's preceding all 3 A's?

N O R M A K L H
N O M A
O A

First, write the M's and A's:

$$\wedge M \wedge M \wedge A \wedge A \wedge A \wedge.$$ (5)

- Put 9 balls into the 6 spaces around these letters: $\left\langle {6 \atop 9} \right\rangle$ ways.
- Now write the letters N O R K L H, N O, O on these balls (order counts). (5)

This is a MISSISSIPPI problem; ~~the number of ways~~ is $\frac{9!}{2!3!}$

Answer: $\boxed{\left\langle {6 \atop 9} \right\rangle \frac{9!}{2!3!}}$

[10] **7.** How many ways can 13 distinguishable balls be put into 5 distinguishable boxes with at least one box empty?

Let $S_i = \{$all ways to distribute balls with box i empty$\}$.
We want to find $|S_1 \cup S_2 \cup \dots \cup S_5|$. By the principle of exclusion and exclusion, this is

$$(|S_1| + |S_2| + \dots + |S_5|) - (|S_1 \cap S_2| + |S_1 \cap S_3| + \dots) + (|S_1 \cap S_2 \cap S_3| + \dots) - \dots + (|S_1 \cap S_2 \cap S_3 \cap S_4 \cap S_5|)$$

$$= 5 \cdot \left(\text{ways to put 13 balls in 4 boxes}\right) - \binom{5}{2} \cdot \left(\text{ways to put 13 balls in 3 boxes}\right) + \dots + \binom{5}{5}\left(\text{ways to put 13 balls in 0 boxes}\right)$$

$$= \boxed{5 \cdot 4^{13} - \binom{5}{2} 3^{13} + \binom{5}{3} \cdot 2^{13} - \binom{5}{4} 1^{13} + 0}$$

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Alternative solution to #5: Let $S_i = \{$all words in which the i^{th} and $(i+1)^{st}$ letters are identical$\}$, for $i = 1$ to 9. Then we want all words which are not in $|S_1 \cup S_2 \cup \dots \cup S_9|$. By P.I.E., this is

$$\boxed{26^{10} - \binom{9}{1} 26^9 + \binom{9}{2} 26^8 - \binom{9}{3} 26^7 + \dots - \binom{9}{9} 26^1.}$$

(Can you see why this is the same answer as that given above, $26 \cdot 25^9$?)

8. There are two rabbits in a colony in year 0, and 7 rabbits in year 1. Each year afterwards, the increase in the number of rabbits is twice the increase of the year before.

a. Write a recurrence relation for P_n, the number of rabbits in year n.

[5] 5 $(P_n - P_{n-1}) = 2(P_{n-1} - P_{n-2})$, or

$P_n = P_{n-1} + 2P_{n-1} - 2P_{n-2}$, or $P_n = 3P_{n-1} - 2P_{n-2}$.

[5] **b.** Solve the recurrence relation.

The characteristic equation is 1 $r^2 = 3r - 2$, or $r^2 - 3r + 2 = 0$,

1 So $r = 2$ or $r = 1$. Thus $P_n = A\,2^n + B \cdot 1^n = A2^n + B$. 1

Since $P_0 = 2$, then $2 = A + B$
Since $P_1 = 7$, then $7 = 2A + B$. So $A = 5$ 1 and $B = -3$. 1

Thus $\boxed{P_n = 5 \cdot 2^n - 3}$

(Check: $P_0 = 2$
$P_1 = 7$
$P_2 = 17$
$P_3 = 37$ ...)

[10] **9.** A "binary sequence of length n" is a sequence of n 0's and 1's. Let a_n be the number of binary sequences of length n which do not contain any consecutive 0's. Find, with explanation, a recurrence relation for a_n.

Each binary sequence of length n must end in either 0 or 1. Call the sequences "type I" or "type II" accordingly.

4 I If a sequence of length n with no consecutive 0's ends in 1, then the sequence of the first $n-1$ numbers also contains no consecutive zeros. Conversely, any sequence of length $n-1$ with no consecutive zeros gives a sequence of ~~length~~ type I when you put a 1 at the end. So the number of sequences of type I is a_{n-1}.

4 II A sequence of type II must end in "10" rather than "00". The $n-2$ numbers preceding the "10" must not have any consecutive zeros. Conversely, any sequence of length $n-1$ with no consecutive zeroes gives a sequence of type II when you add a "10" to the end. So the number of sequences of type II is a_{n-2}.

It follows that $\boxed{a_n = a_{n-1} + a_{n-2}}$ 2