Math 5463
Final exam — solutions

4. Suppose $K(x,y)$ is defined for $(x,y) \in \mathbf{R}^2$, and $K \in L^2(\mathbf{R}^2)$. Suppose $f \in L^2(\mathbf{R})$, and define $g(x)$ for $x \in \mathbf{R}$ by

$$g(x) = \int_{-\infty}^{\infty} K(x,y) f(y)\, dy.$$

Prove that

$$\|g\|_{L^2(\mathbf{R})} \leq \|K\|_{L^2(\mathbf{R}^2)} \|f\|_{L^2(\mathbf{R})}.$$

Soln. Using the Cauchy-Schwarz inequality and Fubini's theorem gives

$$\begin{aligned}
\|g\|^2 &= \int_{-\infty}^{\infty} \left| \int_{-\infty}^{\infty} K(x,y) f(y)\, dy \right|^2 dx \\
&\leq \int_{-\infty}^{\infty} \left[\left(\int_{-\infty}^{\infty} |K(x,y)|^2\, dy \right)^{1/2} \left(\int_{-\infty}^{\infty} |f(y)|^2\, dy \right)^{1/2} \right]^2 dx \\
&= \|f\|^2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |K(x,y)|^2\, dy\, dx = \|f\|^2 \|K\|^2.
\end{aligned}$$

5. Prove that $f(x) = \sqrt{x}$ is absolutely continuous on $[0,1]$.

Soln. It suffices to show that $f(x) = \int_0^x f'(t)\, dt$ for $x \in [0,1]$. For $t > 0$ we have $f'(t) = 1/(2\sqrt{t})$, which is Riemann integrable on $[a,b]$ for any $0 < a < b$, so by the theory of the Riemann integral we have $f(b) - f(a) = \int_a^b f'(t)\, dt$ whenever $0 < a < b$. Also, from the Monotone Convergence Theorem it follows that for all $x \in [0,1]$,

$$\int_0^x f'(t)\, dt = \int_0^x \frac{1}{2\sqrt{t}}\, dt = \lim_{\epsilon \to 0} \int_\epsilon^x \frac{1}{2\sqrt{t}}\, dt.$$

Hence

$$\int_0^x f'(t)\, dt = \lim_{\epsilon \to 0} \int_\epsilon^x f'(t)\, dt = \lim_{\epsilon \to 0} (f(x) - f(\epsilon)) = f(x).$$

6. Suppose $\{f_n\}$ is a (DECREASING) sequence of (NON-NEGATIVE) functions in $L^p(\mathbf{R}^n)$ such that for all $g \in L^{p'}$,

$$\lim_{n \to \infty} \int_{\infty}^{\infty} f_n g\, dx = 0.$$

Prove that f_n converges in measure to the zero function in L^p.

Soln. Let $\eta > 0$ be given, and define $E_n = \{x : f_n(x) > \eta\}$; we want to show that $\lim |E_n| = 0$. Since $\{f_n\}$ is a decreasing sequence, we have $E_1 \supset E_2 \supset E_3 \ldots$. Also, since

$$|E_1| \eta^p \leq \int_{E_1} |f_1|^p < \infty,$$

then $|E_1| < \infty$. Therefore $\lim |E_n| = |E|$, where $E = \cap E_n$.

Let $g = \chi_E$. Then $g \in L^{p'}$ (since $|E| < \infty$), so $\lim \int f_n g = 0$. Also, since f_n is decreasing, we know $f(x) = \lim f_n(x)$ exists for all x; and clearly $f(x) \geq \eta$ for all $x \in E$. Therefore, using the Monotone Convergence Theorem we get

$$0 = \lim \int f_n g = \int f g = \int_E f \geq \int_E \eta = \eta |E|,$$

which implies $|E| = 0$.

(NOTE: the problem as originally stated, without the capitalized corrections, is false: a counterexample is obtained in $\mathbf{R}^1$ by taking f_n to be the characteristic function of the interval $[n, n+1]$.)

7. (ASSUME f, g, c_k, and d_k ARE REAL.) Suppose $\{\phi_k\}$ is an orthonormal basis for L^2, and for $f \in L^2$ and $g \in L^2$, define $c_k = \langle f, \phi_k \rangle$ and $d_k = \langle g, \phi_k \rangle$. Show that

$$\langle f, g \rangle = \sum_k c_k d_k.$$

(Hint: use Parseval's identity.)

Soln. By Parseval's identity,

$$\|f+g\|^2 = \sum |c_k + d_k|^2 = \sum |c_k|^2 + \sum |d_k|^2 + 2 \sum c_k d_k$$

and

$$\|f-g\|^2 = \sum |c_k - d_k|^2 = \sum |c_k|^2 + \sum |d_k|^2 - 2 \sum c_k d_k.$$

Subtracting gives

$$4 \sum c_k d_k = \|f+g\|^2 - \|f-g\|^2 = (\|f\|^2 + \|g\|^2 + 2\langle f, g \rangle) - (\|f\|^2 + \|g\|^2 - 2\langle f, g \rangle) = 4\langle f, g \rangle.$$

(NOTE: If you don't assume the functions involved are real, and replace $\sum c_k d_k$ by $\sum c_k \bar{d}_k$, then to do the problem you have to consider $\|f + ig\|^2$ and $\|f - ig\|^2$ as well. See the text, p. 140.)

8. Define the measure μ on the σ-algebra of Lebesgue measurable subsets of $\mathbf{R}$ by

$$\mu(E) = \begin{cases} 1 & \text{if } 0 \in E \\ 0 & \text{if } 0 \notin E. \end{cases}$$

a. Evaluate (with proof) the integral $g(x) = \int_{-\infty}^{x} 1 \, d\mu$. (It has different values for different choices of x in $\mathbf{R}$.)

Soln. If $x < 0$, then $\mu(-\infty, x) = 0$, so $\int_{-(\infty,x)} 1 \, d\mu = 0$ by a result from class.

If $x \geq 0$ write $(-\infty, x) = \cup E_j$ where the E_j are disjoint and measurable. Let j_0 be such that $0 \in E_{j_0}$; then $\mu(E_{j_0}) = 1$ and $\mu(E_j) = 0$ for $j \neq j_0$. Therefore $\sum_j \mu(E_j) = 1$. Since, by definition,

$$g(x) = \sup_{(-\infty,x)=\cup E_j} \sum_j 1 \cdot \mu(E_j),$$

it follows that $g(x) = 1$.

b. Answer the following questions, with proof.
(i) Is $g(x)$ of bounded variation on $[-1, 1]$?
(ii) Is $g(x)$ absolutely continuous on $[-1, 1]$?
(iii) Is $g(x)$ singular on $[-1, 1]$?

Soln.
(i) Yes, since $g(x)$ is finite and monotone.
(ii) No, because g is not continuous on [-1,1] (every absolutely continuous function on $[a, b]$ is continuous on $[a, b]$.)
(iii) Yes: since $g'(x) = 0$ whenever $x \neq 0$, it follows that $g' = 0$ a.e. in $[-1, 1]$.