Math 4433
Test 3

1. *(15 points)* Suppose $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$. Suppose that $\lim_{x \to c} f(x) = 0$ and suppose there exists M such that $|g(x)| \leq M$ for all $x \in \mathbb{R}$. (Do not assume $\lim_{x \to c} g(x)$ exists.) Prove that $\lim_{x \to c} f(x)g(x) = 0$.

2. *(15 points)* Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous. Let S be the set defined by $S = \{x \in \mathbb{R} : f(x) > 0\}$. Show that S is open.

3. *(15 points)* Let D be a compact subset of $\mathbb{R}$ and suppose that $f : D \to \mathbb{R}$ is continuous. Suppose that for all $x \in D$, $f(x) > 0$. Show that there exists a number $b > 0$ such that for all $x \in D$, $f(x) \geq b$.

4. *(20 points)* Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous. Is it possible for $f(\mathbb{R})$ to equal $\{3, 4\}$ (the set containing just the two numbers 3 and 4)? Prove your answer.

5. *(20 points)* Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^2, & \text{if } x \text{ is rational,} \\ 0, & \text{if } x \text{ is irrational.} \end{cases}$$

Prove that f is differentiable at 0, and find $f'(0)$.

6a. *(10 points)* Use the definition of derivative to show that if $f(x) = \frac{1}{x}$ then for all $c \neq 0$,

$$f'(c) = \frac{-1}{c^2}.$$

6b. *(5 points)* Suppose $g(c) = 3$ and $g'(c) = 7$. Find the derivative of the function $\frac{1}{g(x)}$ at c, using the result of **5a** and the Chain Rule.