3. Let \( \varepsilon > 0 \) be given. By assumption, there exists \( \delta_1 > 0 \) such that if \( x, y \in [0, 1] \) and \( |x - y| < \delta_1 \), then \( |f(x) - f(y)| < \varepsilon \). Also, there exists \( \delta_2 > 0 \) s.t. if \( x, y \in [2, 3] \) and \( |x - y| < \delta_2 \), then \( |f(x) - f(y)| < \varepsilon \).

Define \( \delta = \min(\delta_1, \delta_2, 1) \). Suppose \( x, y \in [0, 1] \cup [2, 3] \) and \( |x - y| < \delta \). Then \( |x - y| < 1 \); so we can't have \( x \in [0, 1] \) and \( y \in [2, 3] \) (because that would mean \( y - x > 2 - 1 = 1 \)) and we can't have \( x \in [2, 3] \) and \( y \in [0, 1] \) (because that would mean \( x - y > 2 - 1 = 1 \)).

So either (i) \( x, y \) are both in \([0, 1]\) or (ii) \( x, y \) are both in \([2, 3]\). In case (i), since \( |x - y| < \delta \leq \delta_1 \), we have \( |f(x) - f(y)| < \varepsilon \); and in case (ii), since \( |x - y| < \delta \leq \delta_2 \), we have \( |f(x) - f(y)| < \varepsilon \).

4. \( \alpha \) If \( x = 0 \), then \( b_n(x) = b_n(0) = 0 \) for all \( n \), so \( \lim b_n(x) = 0 \).

\( \beta \) If \( x > 0 \), then choose \( N \in \mathbb{N} \) s.t. \( N > \frac{x}{2} \). For all \( n \geq N \), we have \( \frac{1}{n} \leq \frac{1}{N} < x \), so \( b_n(x) \to 0 \) (by \( 2 \)). It follows that \( \lim b_n(x) = 0 \).

\( \beta \) \( \lim\left( \int_0^1 b_n(x) \, dx \right) = \lim\left( \int_0^x \left( n^2 x - n^3 x^2 \right) \, dx + \int_x^1 0 \, dx \right) = \lim\left( n^2 \int_0^x x \, dx - n^3 \int_0^x x^2 \, dx \right) = \lim\left( n^2 \left[ \frac{x^2}{2} \right]_0^x - n^3 \left[ \frac{x^3}{3} \right]_0^x \right) = \lim\left( n^2 \cdot \frac{1}{2} - n^3 \cdot \frac{1}{3} \right) = \lim\left( \frac{1}{2} - \frac{1}{3} \right) = \frac{1}{6} \neq 0 \).

5. Since \( b_n \) is continuous on \([0, 1]\) for each \( n \), and \( b_n \) converges pointwise to 0 on \([0, 1]\) by \( 4 \), then if \( b_n \) were to converge uniformly on \([0, 1]\), it would follow from a theorem proved in class that \( \lim s_n b_n = s_n 0 = 0 \). Since \( \lim s_n b_n \neq 0 \) by \( 4 \), then \( b_n \) cannot converge uniformly on \([0, 1]\).
6. We'll prove the statement. Define $g : \mathbb{R} \to \mathbb{R}$ by
   
   $g(x) = \int_0^x f \, dt \quad \text{for } x \in \mathbb{R}.
   
   Let $x \in \mathbb{R}$ be given; we'll show that $g'(x) = f(x)$. Choose an interval $[a, b]$ such that $a < x < b$, then
   
   $g(x) = \int_a^x f \, dt = \int_a^x f + \int_x^b \frac{d}{dx} f
   
   Since $f$ is continuous on $[a, b]$, then by the FTC part 2,
   
   $\frac{d}{dx} \left( \int_a^x f \right) = f(x)$. Hence
   
   $g'(x) = \frac{d}{dx} \int_a^x f + \frac{d}{dx} \int_x^b f = f(x) + 0 = f(x).
   
7. By the FTC part 2, $F'(x) = \sqrt{1 + \sin^2 x} \geq \sqrt{1} = 1 > 0
   
   for all $x > 0$. It follows from a theorem proved in class (actually, in a homework problem) that $F$ is strictly increasing on $[0, \pi]$.}

8. a) If $x = 0$, then $b_n(x) = 2 - n \cdot 0 = 2^0 = 1$ for all $n$, so $\lim b_n(x) = 1$.
   
   If $x \neq 0$, then
   
   $2^{-nx^2} = \frac{1}{(2^{x^2})^n} = \frac{b^n}{b^n} = b^n$
   
   where
   
   $b = \frac{1}{2^{x^2}} \leq \frac{1}{2^0} = 1$, so $\lim b^n = 0$. Thus $\lim b_n(x) = 0$.
   
   This shows that $\lim b_n(x) = b(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0. \end{cases}$

b) No, $b_n$ cannot converge uniformly on $[-1, 1]$; because if $b_n$ were to converge uniformly to $b$ on $[-1, 1]$, then since each $b_n$ is continuous on $[-1, 1]$, it would follow that $b$ is continuous on $[-1, 1]$. But the function $b$ we found in part a) is not continuous on $[-1, 1]$ (it is not cont. at 0.)