Math 4163
Exam 2

Name: Answer key

**NOTE:** On this exam, you may use any solution formula from class, without having to rederive it.

1. (30 points) Solve Laplace's equation inside a rectangle $0 \leq x \leq L$, $0 \leq y \leq H$, with the boundary conditions

$$u(0, y) = 0, \quad u(L, y) = 0, \quad u(x, 0) = 0, \quad \frac{\partial u}{\partial y}(x, H) = f(x).$$

Write the answer as an infinite series, and express the coefficients in terms of the function $f(x)$.

Write $u(x, y) = P(x) Q(y)$ where \( \frac{P''(x)}{P(x)} = - \lambda = \frac{-Q''(y)}{Q(y)} \),

\( P(0) = 0 \), \( P(L) = 0 \), and \( Q(0) = 0 \). The boundary-value problem

\[
\begin{align*}
P''(x) &= -\lambda P(x) \\
P(0) &= 0 \\
P(L) &= 0
\end{align*}
\]

has solutions $P(x) = \sin \left( \frac{n\pi x}{L} \right).$ The equation $Q''(y) = (\lambda) Q(y) = \left( \frac{n\pi}{L} \right)^2 Q(y)$ has solutions $Q(y) = A \cosh \left( \frac{n\pi y}{L} \right) + B \sinh \left( \frac{n\pi y}{L} \right).$ and from $Q(0) = 0$ we get $A = 0$, so $Q(y) = B \sinh \left( \frac{n\pi y}{L} \right).$ Therefore, separated solutions are $u_n(x, y) = B_n \sin \left( \frac{n\pi x}{L} \right) \sinh \left( \frac{n\pi y}{L} \right); \quad n = 1, 2, \ldots$

Write $u(x, y) = \sum_{n=1}^{\infty} B_n \sin \left( \frac{n\pi x}{L} \right) \sinh \left( \frac{n\pi y}{L} \right).$ Then

\[
\begin{align*}
\frac{\partial u}{\partial y}(x, y) &= \sum_{n=1}^{\infty} B_n \sin \left( \frac{n\pi x}{L} \right) \cdot \left( \frac{n\pi}{L} \right) \cosh \left( \frac{n\pi y}{L} \right), \\
\end{align*}
\]

\[
\begin{align*}
\frac{\partial u}{\partial y}(x, H) &= \sum_{n=1}^{\infty} B_n \left( \frac{n\pi}{L} \right) \cosh \left( \frac{n\pi H}{L} \right) \sin \left( \frac{n\pi x}{L} \right) = f(x). \\
\end{align*}
\]

Therefore $B_n \left( \frac{n\pi}{L} \right) \cosh \left( \frac{n\pi H}{L} \right) = \frac{2}{L} \int_0^L f(w) \sin \left( \frac{n\pi w}{L} \right) \, dw$, or

\[
B_n = \frac{2}{L \left( \frac{n\pi}{L} \right) \cosh \left( \frac{n\pi H}{L} \right)} \int_0^L f(w) \sin \left( \frac{n\pi w}{L} \right) \, dw. 
\]
2. (15 points)

a. Suppose \( u(x, y, t) \) is a solution of the equation \( \frac{\partial u}{\partial t} = \nabla^2 u \) on the rectangle \( 0 \leq x \leq 2, \quad 0 \leq y \leq 3 \). Suppose that at all times \( t \), \( u \) satisfies the boundary conditions

\[
\frac{\partial u}{\partial x}(0, y, t) = y, \quad \frac{\partial u}{\partial x}(2, y, t) = 0, \quad \frac{\partial u}{\partial y}(x, 0, t) = 0, \quad \frac{\partial u}{\partial y}(x, 3, t) = 0.
\]

(See diagram.) Find the derivative with respect to \( t \) of the integral

\[
\int_0^3 \int_0^2 u \, dx \, dy.
\]

(Hint: use the divergence theorem.)

\[
\frac{d}{dt} \int_0^3 \int_0^2 u \, dx \, dy = \int_0^3 \int_0^2 \frac{\partial u}{\partial t} \, dx \, dy = \int_0^3 \int_0^2 \nabla \cdot (\nabla u) \, dx \, dy = \oint \nabla u \cdot \hat{n}.
\]

where the integral \( \oint \) is over the boundary of the rectangle and \( \hat{n} \) is the outward normal to the rectangle. Then

\[
\oint \nabla u \cdot \hat{n} = \int_{\text{left edge}} \nabla u \cdot (-\hat{t}) + \int_{\text{top edge}} \nabla u \cdot \hat{j} + \int_{\text{right edge}} \nabla u \cdot (\hat{t} + \hat{j}) + \int_{\text{bottom edge}} \nabla u \cdot (-\hat{j}) =
\]

\[
= \int_0^3 \frac{\partial y}{\partial x} \, dy + \int_0^2 \frac{\partial y}{\partial x} (x, 3) \, dx + \int_0^3 \frac{\partial y}{\partial x} (2, y) \, dy + \int_0^3 \frac{\partial y}{\partial x} (x, 0) \, dx =
\]

\[
= \int_0^3 (-y) \, dy + \int_0^2 0 \, dx + \int_0^3 0 \, dy + \int_0^3 0 \, dx = -\frac{9}{2}, \quad \text{So} \quad \frac{d}{dt} \int_0^3 \int_0^2 u \, dx \, dy = -\frac{9}{2}.
\]

b. Does the equation \( \nabla^2 u = 0 \) have a solution on the rectangle which satisfies the boundary conditions

\[
\frac{\partial u}{\partial x}(0, y) = y, \quad \frac{\partial u}{\partial x}(2, y) = 0, \quad \frac{\partial u}{\partial y}(x, 0) = 0, \quad \frac{\partial u}{\partial y}(x, 3) = 0?
\]

Why or why not?

No, because if \( \nabla^2 u = 0 \), then \( u(x, y) \) is a time-independent solution of \( \frac{du}{dt} = \nabla^2 u \), with \( \frac{du}{dt} = 0 \). But from part a) it would follow that

\[
\frac{d}{dt} \int_0^3 \int_0^2 u \, dx \, dy = \int_0^3 \int_0^2 \frac{\partial u}{\partial t} \, dx \, dy = -\frac{9}{2}, \quad \text{if} \ u \ \text{satisfied the above boundary conditions. Since} \ S_0^3 \int_0^2 \frac{du}{dt} \, dy = \int_0^2 \int_0^3 \frac{d}{dt} \, dx \, dy = 0,
\]

this is impossible.
Problem 2a, alternate solution:

Use the law of conservation of energy:

\[ \frac{d}{dt} \int_0^1 \int_0^2 e \, dx \, dy = - \int_{\partial \Omega} \phi \cdot \mathbf{n} \, dS. \]

Since \( e = c_p u \) and \( \phi = -K_0 \nabla u \), this gives

\[ \frac{d}{dt} \int_0^1 \int_0^2 c_p u \, dx \, dy = K_0 \int_{\partial \Omega} \phi \cdot \mathbf{n} \, dS, \]

and so

\[ \frac{d}{dt} \int_0^1 \int_0^2 u \, dx \, dy = \frac{K_0}{c_p} \int_{\partial \Omega} \phi \cdot \mathbf{n} \, dS = k \int_{\partial \Omega} \phi \cdot \mathbf{n} \, dS. \]

In this problem, \( \frac{d}{dt} u = k \nabla^2 u = \nabla^2 u \), so \( k = 1 \). Therefore

\[ \frac{d}{dt} \int_0^1 \int_0^2 u \, dx \, dy = \int_{\partial \Omega} \phi \cdot \mathbf{n} \, dS. \]

From here we proceed as in the solution on the preceding page: we use the boundary conditions on \( u \) to find that \( \int_{\partial \Omega} \phi \cdot \mathbf{n} \, dS = -\frac{9}{2} \), so

\[ \frac{d}{dt} \int_0^1 \int_0^2 u \, dx \, dy = -\frac{9}{2}. \]
3. (25 points) Consider the problem of solving Laplace’s equation inside a 60° wedge of radius \( a \), subject to the boundary conditions

\[
    u(r, 0) = 0, \quad u(r, \frac{\pi}{3}) = 0, \quad u(a, \theta) = f(\theta),
\]

and the condition that \( u \) remain bounded near \( r = 0 \).

a. Separated solutions of Laplace’s equations in polar coordinates have the form

\[
    u(r, \theta) = G(r)\phi(\theta),
\]

where \( \frac{d^2\phi}{d\theta^2} = -\lambda \phi \). Given that \( u \) satisfies \( u(r, 0) = 0 \) and \( u(r, \frac{\pi}{3}) = 0 \), find the eigenvalues \( \lambda \) and corresponding eigenfunctions \( \phi(\theta) \). You may assume \( \lambda > 0 \).

Since \( u(r, 0) = u(r, \frac{\pi}{3}) = 0 \), then \( \phi(\theta) = A \cos(\sqrt{\lambda} \, \theta) + B \sin(\sqrt{\lambda} \, \theta) \).

If \( \phi(0) = \phi(\frac{\pi}{3}) = 0 \), then \( \phi(0) = \phi(\frac{\pi}{3}) = 0 \). But

\[
    \phi(0) = 0 \Rightarrow 0 = A \Rightarrow \phi(\theta) = B \sin(\sqrt{\lambda} \, \theta),
\]

and

\[
    \phi(\frac{\pi}{3}) = 0 \Rightarrow 0 = B \sin(\sqrt{\lambda} \, \frac{\pi}{3}).
\]

For an eigenvalue we must have

\[
    \sin(\sqrt{\lambda} \, \frac{\pi}{3}) = 0; \quad \text{so} \quad \sqrt{\lambda} \, \frac{\pi}{3} = n\pi \quad (n = 1, 2, 3, \ldots); \quad \text{so} \quad \sqrt{\lambda} = 3n \quad \text{or} \quad \lambda = (3n)^2.
\]

Then \( \phi(\theta) = \sin(3n\theta) \).

b. The equation for \( G(r) \) is \( r^2\frac{d^2G}{dr^2} + r\frac{dG}{dr} - \lambda G = 0 \), which has solutions \( G = r^{3n} \) and \( G = r^{-3n} \). Which of these solutions do we use in this problem, and why?

Here \( \sqrt{\lambda} = 3n \) and the solutions are \( G = r^{3n} \) and \( G = r^{-3n} \). But \( u = G \cdot \phi \) must remain bounded near \( r = 0 \). Since \( r^{3n} \) remains bounded near \( r = 0 \) and \( r^{-3n} \) does not, we use only the solution \( G = r^{3n} \).

c. Write the solution \( u \) of the boundary-value problem as a sum of separated solutions, and express the coefficients in terms of the function \( f(\theta) \).

Take \( u(r, \theta) = \sum_{n=1}^{\infty} B_n r^{3n} \sin(3n\theta) \), so

\[
    u(a, \theta) = \sum_{n=1}^{\infty} B_n a^{3n} \sin(3n\theta) = f(\theta), \quad \text{for} \quad 0 < \theta < \frac{\pi}{3}.
\]

Taking \( L = \frac{\pi}{3} \), we see that \( \frac{n\pi}{L} \theta = 3n\theta \), so the series for \( f(\theta) \) can be rewritten as

\[
    u(a, \theta) = \sum_{n=1}^{\infty} B_n a^{3n} \sin(n\pi \theta) = f(\theta), \quad \text{for} \quad 0 < \theta < L.
\]

It is a Fourier sine series for \( f(\theta) \) on \([0, L]\), so form the formula for coefficients of a Fourier sine series, we get

\[
    B_n a^{3n} = \frac{2}{L} \int_{0}^{L} f(\omega) \sin(n\pi \omega) \, d\omega
\]

or

\[
    B_n = \frac{2}{(m\pi)a^{3n}} \int_{m\pi/3}^{\pi/3} f(\omega) \sin(3n\omega) \, d\omega.
\]
4. (30 points) Consider the function \( f(x) \) defined on \([0, \pi]\) by

\[
f(x) = \begin{cases} 
1 & \text{for } 0 \leq x \leq \frac{\pi}{2} \\
0 & \text{for } \frac{\pi}{2} \leq x \leq \pi.
\end{cases}
\]

a. Find the coefficients of the Fourier cosine series for \( f \) on \([0, \pi]\). Write out the first three non-zero terms of the series.

\[
A_0 = \frac{1}{\pi} \int_0^\pi f(x) \, dx = \frac{1}{\pi} \int_0^{\frac{\pi}{2}} 1 \, dx = \frac{1}{\pi} \cdot \frac{\pi}{2} = \frac{1}{2}.
\]

For \( n \geq 1 \), \( A_n = \frac{2}{\pi} \int_0^\pi f(x) \cos(nx) \, dx = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \cos(nx) \, dx = \frac{2}{\pi} \left[ \frac{\sin(nx)}{n} \right]_0^{\frac{\pi}{2}} = \frac{2}{\pi n} \sin \left( \frac{n\pi}{2} \right).
\]

Thus \( A_1 = \frac{2}{\pi} \), \( A_2 = 0 \), \( A_3 = -\frac{2}{3\pi} \), \( A_4 = 0 \), etc.

We have

\[
f(x) \sim A_0 + \sum_{n=1}^{\infty} A_n \cos(nx) = \frac{1}{2} + \frac{2}{\pi} \cos x - \frac{2}{3\pi} \cos 3x + \ldots
\]

b. Sketch the graph of the function to which the series in part a converges. Show the function on at least the interval \([-2\pi, 2\pi]\). Use an \( \times \) to mark the values of the function at the points where it is discontinuous.

(The series converges to the \underline{even extension} of \( f \) which is \underline{periodic} of \underline{period} \( 2\pi \).)

\( \times \)'s in correct spots: \(3\) p.t. \(4\) p.t