Answers to problems on Exam 1

1. [5] Suppose $S \subseteq \mathbb{R}$. We say $u \in \mathbb{R}$ is a <u>supremum of S</u> if
(i) u is an upper bound of S, and
(ii) for every upper bound v of S, $u \le v$

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2. [5] (a.) Yes; by the Completeness Axiom, since we know S is bounded above by 0.

[5] (b.) Yes; we know 0 is an upper bound for S, so by part (ii) of the definition of supremum, $\sup S \le 0$.

[5] (c.) No. For example, if $S = (-\infty, 0)$, then it is true that for all $x \in S$, $x < 0$. But in this case $\sup S = 0$ (as proved in class), so $\sup S < 0$ is false.

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3. [5] (a.) Suppose (x_n) is a sequence and $L \in \mathbb{R}$. We say L is a <u>limit of (x_n)</u> if, for every $\varepsilon > 0$, there exists $K \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, if $n \ge K$ then $|x_n - L| < \varepsilon$.

[20] (b.) Let $\varepsilon > 0$ be given. There exists $K_1 \in \mathbb{N}$ such that if $n \ge K_1$ then $|x_n - x| < \frac{\varepsilon}{2}$; and there exists $K_2 \in \mathbb{N}$ such that if $n \ge K_2$ then $|y_n - y| < \frac{\varepsilon}{2}$. Define $K = \max\{K_1, K_2\}$.
If $n \ge K$, then $n \ge K_1$ and $n \ge K_2$, so $|x_n - x| < \frac{\varepsilon}{2}$ and $|y_n - y| < \frac{\varepsilon}{2}$.
Hence $|(x_n + y_n) - (x+y)| = |(x_n - x) + (y_n - y)| \le |x_n - x| + |y_n - y| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$,
so $|(x_n + y_n) - (x+y)| < \varepsilon$.

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4. [10]. By Bernoulli's Inequality, if $a > -1$ then for all $n \in \mathbb{N}$, $(1+a)^n \ge 1 + na$. Taking $a=1$ we get that for all $n \in \mathbb{N}$, $2^n \ge 1 + n$. By the Archimedean Property of $\mathbb{R}$, $\mathbb{N}$ is not bounded above, so there exists $n \in \mathbb{N}$ s.t. $n > 10^{1{,}000{,}000} - 1$.
Hence $1 + n > 10^{1{,}000{,}000}$, so $2^n > 10^{1{,}000{,}000}$.

5. [25] Let $\varepsilon > 0$ be given. 2

There exists 2 $K \in \mathbb{N}$ s.t. $K > \left(\frac{7}{\varepsilon}\right) - 7$. 2

If $n \geq K$ 2 then $n > \left(\frac{7}{\varepsilon}\right) - 7$, 2 so $n+7 > \frac{7}{\varepsilon}$, 2 so $\varepsilon > \frac{7}{n+7}$. 4

Hence $\left| \frac{-7}{n+7} \right| < \varepsilon$, 2

so $\left| \left(\frac{n}{n+7}\right) - \left(\frac{n+7}{n+7}\right) \right| < \varepsilon$, 2

so $\left| \left(\frac{n}{n+7}\right) - 1 \right| < \varepsilon$. 5

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6. [20] Let $\varepsilon > 0$ be given. 2

There exists $K \in \mathbb{N}$ s.t. $K > \frac{1}{\varepsilon}$. 2

If $n \geq K$ Then $n > \frac{1}{\varepsilon}$, so $\frac{1}{n} < \varepsilon$. 2

Also, $\frac{n!}{n^n} = \frac{n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1}{n \cdot n \cdot n \cdots n \cdot n \cdot n}$ 2 (where there are n factors of "n" in the denominator)

and $\frac{n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1}{n \cdot n \cdot n \cdots n \cdot n \cdot n} \leq \frac{n \cdot n \cdot n \cdots n \cdot n \cdot 1}{n \cdot n \cdot n \cdots n \cdot n \cdot n}$ 4 (where there are $n-1$ factors of "n" in the numerator and n factors of "n" in the denominator)

so $\frac{n(n-1)(n-2)\cdots 3 \cdot 2 \cdot 1}{n \cdot n \cdot n \cdots n \cdot n \cdot n} \leq \frac{1}{n}$, 2

and hence $\frac{n!}{n^n} \leq \frac{1}{n}$. 2

Therefore $\frac{n!}{n^n} < \varepsilon$, and so 2

$\left| \frac{n!}{n^n} - 0 \right| < \varepsilon$. 2