Math 4443/5443
Exam 1

Instructions: Do problems 1, 2 and any five of the six remaining problems (for a total of 100 possible points).

1. (20 points)
   a. Give the definition of “Riemann integral of a function on $[a, b]$”.
   b. Prove that a function $f$ can have no more than one Riemann integral on $[a, b]$.

2. (20 points) Prove that if $f$ is differentiable on $[a, b]$ and $f'(x) = 0$ for all $x \in [a, b]$, then $f$ is constant on $[a, b]$.

3. (12 points) Prove that if $f$ and $g$ are differentiable on $\mathbb{R}$, $f(0) = g(0) = 0$, and $f'(x) \leq g'(x)$ for all $x > 0$, then $f(x) \leq g(x)$ for all $x > 0$.

4. (12 points) Prove that $e^x - e^{-x} \geq 2x$ for all $x > 0$ (you may use problem 3, even if you haven’t proved it).

5. (12 points) Suppose $f''(x)$ exists for all $x \in \mathbb{R}$, and $f''(x) > 0$ for all $x \in \mathbb{R}$. Suppose also that $f(0) = f'(0) = 0$. Prove that $f(x) > 0$ for all $x \in \mathbb{R}$. (The easiest way to do this is to use Taylor’s theorem.)

6. (12 points) Prove that if $f \in \mathcal{R}[a, b]$, then $\left| \int_a^b f \, dx \right| \leq \int_a^b |f| \, dx$.

7. (12 points) Either prove that the following statement is true, or give a counterexample to show that it is false:
   “If $f \in \mathcal{R}[a, b]$ and $\int_a^b f = 0$, then $f(x) = 0$ for all $x \in [a, b]$.”

8. (12 points) Suppose $f : [a, b] \to \mathbb{R}$, and suppose there exist sequences of tagged partitions $\{\mathcal{P}_n\}$ and $\{\mathcal{Q}_n\}$ such that $\|\mathcal{P}_n\| < 1/n$ and $\|\mathcal{Q}_n\| < 1/n$ for all $n$, and

$$\left| S(f, \mathcal{P}_n) - S(f, \mathcal{Q}_n) \right| \geq 1$$

for all $n$. Show that $f \notin \mathcal{R}[a, b]$. (You may use any theorem from class.)