Shawndrea Miller

Pythagorean Triples

A Pythagorean triple consists of three positive integers a, b, and c, such that $a^2 + b^2 = c^2$. Such a triple is commonly written (a, b, c). The smallest and best-known Pythagorean triple is $(a, b, c) = (3, 4, 5)$. If (a, b, c) is a Pythagorean triple, then so is (ka, kb, kc) for any positive integer k. The name is derived from the Pythagorean theorem, of which every Pythagorean triple is a solution. The converse is not true. For instance, the triangle with sides a = b = 1 and c = √2 is right, but (1, 1, √2) is not a Pythagorean triple because √2 is not an integer. Moreover, 1 and √2 do not have an integer common multiple because √2 is irrational.

There are 16 primitive Pythagorean triples with $c \leq 100$:

(3, 4, 5) (5, 12, 13) (7, 24, 25) (8, 15, 17) (9, 40, 41) (11, 60, 61) (12, 35, 37)

(13, 84, 85) (16, 63, 65) (20, 21, 29) (28, 45, 53) (33, 56, 65) (36, 77, 85) (39, 80, 89)

(48, 55, 73) (65, 72, 97)

This following formula was given by Euclid (c. 300 B.C.) in his book *Elements* and is often referred to as Euclid's formula. Euclid is known as the "Father of Geometry" and his book *Elements* is supposedly the most successful text book in the history of mathematics. It goes that

$a = (m^2 - n^2)\ :\ b = (2mn)\ :\ c = (m^2 + n^2)$ where m and n are two positive integers with m > n. The triple generated will be primitive if and only if m and n are coprime and exactly one of them is even (if both n and m are odd, then a, b, and c will be even, and so the Pythagorean triple will not be primitive). Not all non-primitive Pythagorean triples can be generated with this formula, but every primitive triple (possibly after exchanging a and b) arises in this fashion from a unique pair of coprime numbers m, n. This shows that there are infinitely many primitive Pythagorean triples.

The following shows that Euclid's Formula does in fact work:

$a^2 + b^2 = c^2 \rightarrow (m^2 - n^2)^2 + (2mn)^2 = (m^2 + n^2)^2 \rightarrow m^4 - 2m^2n^2 + n^4 + 4m^2n^2 = m^4 + 2m^2n^2 + n^4$

- $4m^2n^2 = 4m^2n^2$.

Ex(s): m = 2, n = 1 (3,4,5)

$(2^2 - 1^2)^2 + (2*2*1)^2 = (2^2 + 1^2)^2 \rightarrow (3)^2 + (4)^2 = (5)^2 \rightarrow 9 + 16 = 25$

m = 3, n = 2 (5,12,13)

$(3^2 - 2^2)^2 + (2*3*2)^2 = (3^2 + 2^2)^2 \rightarrow (5)^2 + (12)^2 = (13)^2 \rightarrow 25 + 144 = 169$

m = 4, n = 3 (7,24,25)

$(4^2 - 3^2)^2 + (2*4*3)^2 = (4^2 + 3^2)^2 \rightarrow (7)^2 + (24)^2 = (25)^2 \rightarrow 49 + 576 = 625$

Some properties of primitive Pythagorean triples include:

- Exactly one of *a*, *b* is odd; *c* is odd.
- The area ($A = ab/2$) is an integer.
- Exactly one of *a*, *b* is divisible by 3.
- Exactly one of *a*, *b* is divisible by 4.
- Exactly one of *a*, *b*, *c* is divisible by 5.
- Exactly one of *a*, *b*, $(a + b)$, $(b - a)$ is divisible by 7.
- All prime factors of *c* are primes of the form 4n+1.
- At most one of *a*, *b* is a square.
- Every integer greater than 2 that is not congruent to 2 mod 4 is part of a primitive Pythagorean triple. Examples of integers not part of a primitive pythagorean triple: 6,10,14,18
- Every integer greater than 2 is part of a primitive or non-primitive Pythagorean triple, for example, the integers 6,10,14, and 18 are not part of primitive triples, but are part of the non-primitive triples 6,8,10; 14,48,50 and 18,80,82.
- There exist infinitely many Pythagorean triples whose hypotenuses are squares of natural numbers.
- There exist infinitely many Pythagorean triples in which one of the legs is the square of a natural number.
- There exist infinitely many Pythagorean triples in which the hypotenuse and the longer of the two legs differ by exactly one.
- There exist infinitely many Pythagorean triples in which the hypotenuse and the longer of the two legs differ by exactly two.
- There are no primitive Pythagorean triples in which the hypotenuse and a leg differ by a prime number greater than 2.
- For each natural number *n*, there exist *n* Pythagorean triples with different hypotenuses and the same area.
- For each natural number *n*, there exist at least *n* different Pythagorean triples with the same leg *a*, where *a* is some natural number
- For each natural number *n*, there exist at least *n* different triangles with the same hypotenuse.
- In every Pythagorean triple, the radius of the incircle and the radii of the three excircles are natural numbers. (Actually the radius of the incircle can be shown to be $r = n(m - n)$)
- There is no Pythagorean triple in which the hypotenuse and one leg are the legs of another Pythagorean triple.
- In a pythagorean triplet a+b=c+2[(c-a)(c-b)/2]^1/2.
- (c-a)(c-b)/2 is always a perfect square.

If $a^2 + b^2 = c^2$ is a primitive Pythagorean triple, where *a* is odd, then

$$\frac{c+a}{b} = \frac{m}{n},$$

$$\frac{c+b+a}{c+b-a} = \frac{m}{n}$$

$b/(c-a) = \frac{m}{n}$

$(a+c-b)/(a+b-c) = \frac{m}{n}$ where each fraction is reduced to lowest terms and $m > n$.

For example, with the (3,4,5) triple, m = 2 and n = 1:

- (c + a)/b = m/n → (5+3)/4 = 2/1 → 2=2
- (c + b + a)/ (c + b – a) = m/n → (5+4+3)/(5+4-3) = (2/1) → (12)/(6) = (2)/(1) → 2=2
- b/ (c-a) = m/n → (4)/(5-3) = (2)/(1) → (4)/(2)=(2)/(1) → 2=2
- (a + c - b)/(a + b – c) = m/n → (3+5-4)/(3+4-5) = (2)/(1) → (4)/(2)=(2)/(1) → 2=2

And with the (5,12,13) triple, m = 3 and n = 2:

- (c + a)/b = m/n → (13+5)/(12)=(3/2) → (18/12) = (3/2) → (3/2)=(3/2)
- (c + b + a)/ (c + b – a) = m/n →(13+12+5)/(13+12-5)=(3/2) → (30/20)=(3/2) → (3/2)=(3/2)
- b/ (c-a) = m/n →(12)/(13-5) = (3/2) → (12/8)=(3/2) → (3/2)=(3/2)
- (a + c - b)/(a + b – c) = m/n →(5+13-12)/(5+12-13)=(3/2) → (6/4)=(3/2) → (3/2)=(3/2).

Sources:

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http://jwilson.coe.uga.edu/EMT668/EMT668.Folders.F97/Edenfield/Pythtriples/Pythriples.html
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Diana Davis

Fermat's Last Theorem

Pierre de Fermat was a seventeenth century mathematician. Fermat's last theorem is one of the most famous problems in the history of mathematics. This theorem states that the equation $x^n + y^n = z^n$ has no integer solutions for $n > 2$. Fermat stated this theorem in the margins of his book, *Arithmetica,* writing:

> "It is impossible to separate a cube into two cubes, or a fourth power into two fourth powers, or in general, any power higher than the second into two like powers. I have discovered a truly marvelous proof of this, which this margin is too narrow to contain."

Though Fermat claimed to have proved this theorem, a proof was never found, and this problem became one of the most famous unsolved problems. Mathematicians proved many special cases of this theorem. Several incorrect proofs of the general statement have been offered since the time of Fermat, but it wasn't until 1993 that a correct proof was published by Andrew Wiles. This proof is very long and complex, as are most of the special cases. The most simple case of the theorem to prove is when $n = 4$. This was the first case to be proved, and was proved by Fermat himself. So, I will prove that there are no integer solutions to the equation $x^4 + y^4 = z^4$.

This proof uses the method descent, which is a type of induction. The method of descent uses the fact that if a proposition P(n) is true for some positive integer n, then there is a smallest integer for which it is true. However, if P(n) is true for every n implies that P(n) is true for a some smaller integer, than there is no smallest integer. Thus, P(n) is false for every n. The proof that there are no integer solutions for $x^4 + y^4 = z^4$ is as follows:

We will actually consider $x^4 + y^4 = z^2$ and prove that there are no integer solutions to this equation (this is actually a more powerful statement, and surely if there are no solutions with z^2 then there are none with $(z^2)^2 = z^4$.

Proof:

Assume there exists some positive z such that the statement $x^4 + y^4 = z^2$ is true. If this is true, there must be some smallest integer z that satisfies this equation. Let z be an integer for which the statement is true. Then we can assume that gcd(x, y, z) = 1 (x and y are coprime). This implies that z must be odd, and x and y must have opposite parity. Let x be even and y be odd. Since $x^4 + y^4 = z^2$ is equivalent to $(x^2)^2 + (y^2)^2 = z^2$ which is a Pythagorean triple. Thus, we can consider the following equations:

$$x^2 = 2pq, \quad y^2 = q^2 - p^2, \quad z = p^2 + q^2$$

where q and p are positive integers, q > p, and p and q are coprime with opposite parity. By solving the equation with y^2, we have the equation $y^2 + p^2 = q^2$ which is a Pythagorean triple. So, this gives us the following equations:

$$p = 2rs, \quad y = r^2 - s^2, \quad q = r^2 + s^2$$

where r and s are positive integers, r > s, and r and s are coprime with opposite parity. Thus, $\gcd(r^2, s^2) = 1$. From this, it follows that $\gcd(r^2, r^2 + s^2) = 1 = \gcd(s^2, r^2 + s^2)$, which implies that $\gcd(rs, r^2 + s^2) = 1$. Since $(rs/2)^2 = rs(r^2 + s^2)$, all of r, s, and $r^2 + s^2$ are squares. So we can set

$Z^2 = q$, $X^2 = r$, and $Y^2 = s$ which leads to a new equation,

$$X^4 + Y^4 = Z^2$$

where $Z^2 = q < q^2 + p^2 = z < z^2$. Thus, we have a solution where $Z < z$. These steps can be repeated to produce an infinite amount of smaller solutions that are square sums of two fourth powers. This contradicts the fact that there must be one smallest integer, z, thus the statement is false and there are no integer solutions to the solution (QED).

Sources:

http://planetmath.org/encyclopedia/ExampleOfFermatsLastTheorem.html

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