Complex Analysis

Here are some sample problems from the text by Brown and Churchill. These are less abstract and more basic than most of the problems in Greene and Krantz' text, but I still think they'd be good practice problems for the first exam in this class. The test would probably have a few problems of this caliber mixed in with some which are more like the ones in Greene and Krantz.

Some of the problems below you might be able to do just by looking at them. On others, you might get stuck. If that happens, feel free to discuss it with me (or any of the other students).

1. By factoring $z^4 - 4z^2 + 3$ into two quadratic factors and using the triangle inequality, show that if z lies on the circle $|z| = 2$, then

$$\left|\frac{1}{z^4 - 4z^2 + 3}\right| \leq \frac{1}{3}.$$

2. Prove that two nozero complex numbers z_1 and z_2 have the same moduli if and only if there are complex numbers c_1 and c_2 such that $z_1 = c_1c_2$ and $z_2 = c_1\overline{c_2}$. (Suggestion: first think how the arguments of c_1 and c_2 should be related to those of z_1 and z_2.

3. Find the four zeros of the polynomial $z^4 + 4$, and use them to factor $z^4 + 4$ into two quadratic factors with real coefficients.

4. Write $f(x, y) = x^2 - y^2 - 2y + i(2x - 2xy)$ as a polynomial in z and $\overline{z}$, and simplify the result.

5. Show that the limit of the function

$$f(z) = \left(\frac{z}{\overline{z}}\right)^2$$

as z tends to 0 does not exist.

6. Suppose that $f(z_0) = g(z_0) = 0$ and that $f'(z_0)$ and $g'(z_0)$ exist, where $g'(z_0) \neq 0$. Show that

$$\lim_{z \to z_0} \frac{f(z)}{g(z)} = \frac{f'(z_0)}{g'(z_0)}.$$

(Hint: use the definition of derivative.)

7. Suppose $f(z)$ is the function defined by

$$f(z) = \begin{cases} \overline{z}^2/z & \text{when } z \neq 0, \\ 0 & \text{when } z = 0. \end{cases}$$

Show that (a) $f'(0)$ does not exist, but (b) the Cauchy-Riemann equations $u_x = v_y$ and $u_y = v_x$ are satisfied at $z = 0$.

8. Evaluate $\int_\gamma f(z)\ dz$ if $f(z) = (z + 2)/z$ and γ is (a) the semicircle $z = 2e^{i\theta}(0 \leq \theta \leq \pi)$, (b) the semicircle $z = 2e^{i\theta}(\pi \leq \theta \leq 2\pi)$, (c) the circle $z = 2e^{i\theta}(0 \leq \theta \leq 2\pi)$.

9. Show that if γ is any closed contour which does not pass through the point z_0, and n is any integer such that $n \neq -1$, then $\int_\gamma (z - z_0)^n\ dz = 0$.

10. Suppose $z \in \mathbf{C}$. Show that if γ is any positively oriented simple closed C^1 curve which is deformable in $\mathbf{C}\ \{z\}$ to a circle containing z in its interior, then

$$\int_\gamma \frac{\zeta^3 + 2\zeta}{(\zeta - z)^3}\ d\zeta = 6\pi i z.$$

On the other hand, show that if γ is deformable in $\mathbf{C}\ \{z\}$ to a circle which does not contain z in its interior, then the integral equals 0.