Math capstone presentation
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My team choose the Random Fibonacci Sequence problem.
The problem was broken down to three segments and each was assigned to one team member.

I took the first part, where we introduced the Fibonacci sequence, normal and random. I also used an excel spreadsheet to show the sequences and graphs of the sequences to the class.

First, I explained the history of the Fibonacci numbers to the class, and reiterated how the sequence is generated. I explained how the sequence was used as a model for the growth or rabbits. This growth, however, does not take into account any death and has a perfect birth rate. I displayed to the class a spreadsheet showing the normal Fibonacci sequence taken out to 5000 iterations. In one column I had F(n) / F(n-1) iterated out and had a graph displaying the convergence of the sequence to the golden ratio. In addition I had a column of the golden ratio to the nth root. And finally I had a column highlighting the constant .44721… and how it came about from dividing F(n) by the golden ratio to the nth power.

Second, I introduced the idea of adding an element of randomness to the sequence. Instead of always adding the previous two elements of the sequence, what if sometimes the previous elements were subtracted. Could this be a model to more accurately represent the growth of species in nature? I wrote several functions in Visual Basic to generate random numbers and iterate through the Random Fibonacci sequence. My Random() function generated either a 0 or a 1. I summed the entire sequence of random numbers and got a result of 2469 from 5000 iterations, thus the number of 0's was close to the number of 1's. My Random_Fibonacci () function took as arguments the previous two numbers in the sequence and the random 0 or 1 as the third argument. In the function, if the random number is 0, then the two numbers given are added. If the random number is 1, then the two given numbers were subtracted. Thus, at each iteration in the sequence, the result can either be from addition or subtraction. From the numbers generated from this function I took the absolute value in a new column. I graphed this column to show that the absolute value of the Random Fibonacci sequence grow exponentially, just like the normal Fibonacci numbers. I had another column with the nth root of Random_Fibonacci (n). This sequence very slowly converges to the new constant of 1.13… This is the constant that Viswanath wrote his paper on. I graphed this column as well.

Third, I talked about the constant Beta Star. If we modify the Random Fibonacci recursion equation "F(n+1) = F(n) +- F(n-1)" by multiplying the F(n-1) part by Beta, we can get some interesting results. Beta Star was found to be .70258. With this value included in the equation, the nth root of Random_Fibonacci () converged to 1. This could be seen as the equalizer in the growth or decay of a species. With any number greater than Beta Star, the population grows, with any number less than Beta Star, the population declines.

I wrote a new function with the added Beta Star argument and ran the function four times with different values for Beta. Beta= .70258, Beta = 1, Beta = .01, and Beta = 2. I then took the nth root of these sequences to show that Beta Star (Beta = .70258) converges to 1; Beta = 1 converges to 1.13… just like the orginal Random_Fibonacci() sequence; Beta = .01 decays; and Beta = 2 grows.

My sources were taken from:

http://www.sciencenews.org/pages/sn_arc99/6_12_99/bob1.htm
http://www.ams.org/mcom/2000-69-231/S0025-5718-99-01145-X/home.html#Abstract
http://arxiv.org/abs/math/0510159
http://en.wikipedia.org/wiki/Viswanath's_constant
http://en.wikipedia.org/wiki/Embree-Trefethen_constant
http://mathdl.maa.org/convergence/1/?pa=content&sa=viewDocument&nodeId=630&bodyId=1002

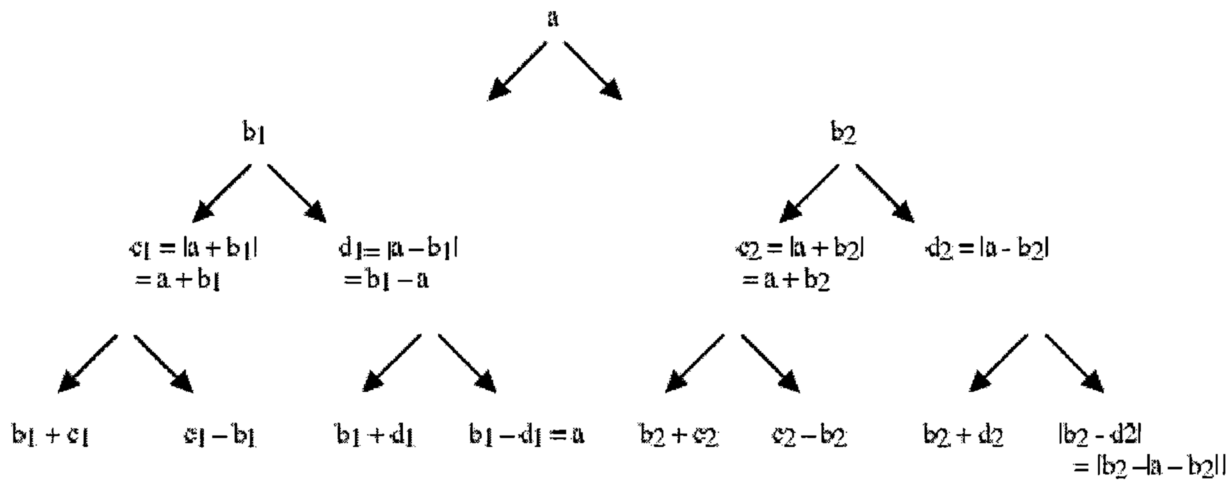
Presentation Write-Up

In 1999, Divakar Viswanath discovered a new constant which emerged from looking at the Fibonacci sequence a bit differently. Traditionally, the Fibonacci sequence recursion relation has F(n) = F(n-1) + F(n-2). The growth of the term F(n) is said to model the population growth of rabbits in a closed system. However, this model is biologically unrealistic because there is no die off, i.e. the rabbits keep multiplying without ever succumbing to predators. Perhaps to create a more realistic model, Viswanath proposed the study of the sequence where F(n) = F(n-1) $\pm$ F(n-2), where the second term has a 50/50 chance of being positive or negative. What he found was that the absolute value of the nth term grew exponentially at a constant rate. This exponential growth is given by the limit as n goes to infinity of the nth root of the nth term, or 1.13198824…. Thus Viswanath found order emerging from a seemingly random sequence.

Viswanath's proof involves an old result from Furstenberg and Kesten concerning the multiplication of random matrices and some clever computational floating point algebra which is beyond the scope of this paper. Therefore in order to understand from where the constant 1.13… emerges, we will turn to a more qualitative argument given by Eran Makover and Jeffrey McGowan. The following ideas come directly from their 2005 publication, *An Elementary Proof that Random Fibonacci Sequences Grow Exponentially.*

First we will imagine the aforementioned sequence as a tree where the first branch, say x(i) begets two leafs, x(i-1) + x(i) and x(i-1) – x(i). Thus, starting with x(i), every possibility is represented. Furthermore, we will take the absolute value of each entry, leaving only positive, possibly reversed, terms. To glean information from this system, we will try and find the expectation value of the absolute value of the nth term. (Note that one would expect the expectation value of the nth term to be approximately zero, thus we take the absolute value.) This expectation value is just the sum of the nth row divided by 2^n.

We will now suppose that the first term in our tree is a. a has two children, b1 and b2, where at least one of the children must be bigger than a. So, without loss of generality, let b1 $\geq$ a. The tree is completed below:



The sum of the last row is given by s = a + b1 + b2 + 2c1 + 2c2 + d1 + d2 + |b2 - |a - b2||. We shall show that this sum is bounded above and below as follows:

$4a + b1 + b2 + c1 + c2 + d1 + d2 \leq s \leq 4a + 2b1 + 2b2 + c1 + c2 + d1 + d2$

In order to see the left hand inequality, we substitute c1 + c2 = 2a + b1 + b2 in our equation for s, giving s = 3a + 2b1 + 2b2 + c1 + c2 + d1 + d2 + |b2 - |a - b2||. Since b1 $\geq$ a, it is clear that the left inequality is correct. Using the same form for s, we can see that in order to prove the right hand inequality, we must show that $|b2 - |a - b2|| \leq a$. We shall do this by considering cases:

1) Assume b2 $\geq$ a. Then $|b2 - |a - b2|| = |b2 - b2 + a| = a$.

2) Assume $a > b2 \geq a/2$. Then $|b2 - |a - b2|| = |b2 - (a - b2)| = |2b2 - a| = 2b2 - a < a$.

3) Assume $b2 \leq a/2$. Then $|b2 - |a - b2|| = |b2 - (a - b2)| = |2b2 - a| = a - 2b2 < a$.

Thus our inequality is correct. Next, note that if we wish to define the sum of the nth row as S[n], then the lower bound may be written as 4S[n-3] +S[n-2] + S[n-1], and the upper bound may be written as 4S[n-3] + 2S[n-2] + S[n-1]. To solve this pair of recursion relations, we make the substitution of $S[n] = r^n$. This leaves solutions which depend of the real roots of $r^3 - r^2 - r - 4$ and $r^3 - r^2 - 2r - 4$, respectively. Finally $r/2^n$ gives the growth rate of S[n], where the factor of $1/2^n$ accounts for the branch splitting.

From this calculation, Makover and McGowan found that $1.12095\ldots \leq E[|x(n)|]^{1/n} \leq 1.23375\ldots$, where E[|x(n)|] is the expectation value of the absolute value of the nth term in the sequence. One can see that Viswanath's constant of 1.13… fits nicely between these bounds.

A more general case of this sequence occurs when a factor of β is added to the second term, i.e. $x(n) = x(n-1) \pm \beta\, x(n-2)$. We can set up this problem is an analogous fashion to the one we just solved. In doing so, Makover and McGowan found that at a value of $\beta = 0.707107$, the sequence may start to decay. A more exact calculation by Embree and Trefethen found that the sequence begins to decay at $\beta = 0.70258$. Any β above this value will result in an exponential growth, e.g. when $\beta = 1$ we get Viswanath's constant of 1.13… for the growth rate.

Works Cited

Makover, Eran and Jeffrey McGowan. An Elementary Proof That Random Fibonacci Sequences Grow Exponentially. arXiv:math/0510159v2 [math.NT]. 28 Oct 2005.

Viswanath, Divakar. Random Fibonacci Sequences and the Number 1.13198824.... Mathematics of Computation. Vol. 69. No. 231. pp 1131 - 1155. 10 Jun 1999.

Embree, Mark and Lloyd Trefethen. Growth and Decay of Random Fibonacci Sequences. *Proceedings: Mathematical, Physical and Engineering Sciences.* Vol. 455. No. 1987. pp. 2471-2485. 08 Jul 1999.