2.5.16 - Solution

(a) Let \( R \) stand for the rectangular region in the plane given by \( 0 \leq x \leq L \) and \( 0 \leq y \leq H \). Let \( \mathcal{S} \) stand for the four line segments which form the boundary of this region.

According to the law of conservation of heat energy (equation (1.5.1) in the text), if there are no sources \( (Q \equiv 0) \) then the temperature \( \mathbf{u} \) within \( R \) and the heat flux vector \( \mathbf{\Phi} \) on \( \mathcal{S} \) satisfy

\[
\frac{d}{dt} \iiint_{R} \rho c u \, dV = \oint_{\mathcal{S}} \mathbf{\Phi} \cdot \mathbf{n} \, ds.
\]

Here \( \rho \) is the specific heat, \( c \) is the density, and \( \mathbf{n} \) is the outward unit vector normal to \( \mathcal{S} \). Since this is a two-dimensional problem, the triple integral in (1.5.1) is replaced by a double integral, and the integral in (1.5.1) is replaced by a line integral over \( \mathcal{S} \), as explained on p. 27 of the text.

In this problem, the temperature is at equilibrium, so \( \frac{du}{dt} \equiv 0 \), and hence

\[
\frac{d}{dt} \iiint_{R} \rho c u \, dV = \iiint_{R} \frac{\partial}{\partial t} \rho c u \, dV = 0.
\]

Therefore, equation (1) gives

\[
\oint_{\mathcal{S}} \mathbf{\Phi} \cdot \mathbf{n} \, ds = 0.
\]

Now by Fourier's law of heat conduction, equation (1.5.7) in the text, we have \( \mathbf{\Phi} = -K_0 \nabla u \), where \(-K_0\)
is the thermal conductivity. Therefore from (2) we get

(3) \( \phi \left( \nabla u \right) \cdot \hat{n} = 0 \).

The line integral in (3) splits into four integrals, one along each of the line segments forming the boundary of \( \Omega \).

On the right boundary, \( \hat{n} = \hat{i} \), so \( \nabla u \cdot \hat{n} = \left( \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} \right) \cdot \hat{i} = \frac{\partial u}{\partial x} \). On the left boundary, \( \hat{n} = -\hat{i} \), so \( \nabla u \cdot \hat{n} = -\frac{\partial u}{\partial x} \).

On the top boundary, \( \hat{n} = \hat{j} \), so \( \nabla u \cdot \hat{n} = \frac{\partial u}{\partial y} \). On the bottom boundary, \( \hat{n} = -\hat{j} \), so \( \nabla u \cdot \hat{n} = -\frac{\partial u}{\partial y} \). Therefore, equation (3) becomes:

\[
0 = \int_0^L -\frac{\partial u}{\partial y} (x,0) \, dx + \int_0^H \frac{\partial u}{\partial x} (L,y) \, dy + \int_0^L \frac{\partial u}{\partial y} (x,H) \, dx + \int_0^H \frac{\partial u}{\partial x} (0,y) \, dy
\]

From the given boundary conditions, \( \frac{\partial u}{\partial y} (x,0) = 0 \), \( \frac{\partial u}{\partial x} (L,y) = g(y) \), \( \frac{\partial u}{\partial y} (x,H) = f(x) \), and \( \frac{\partial u}{\partial x} (0,y) = 0 \), so we get

(4) \( 0 = \int_0^H g(y) \, dy + \int_0^L f(x) \, dx \),

which is our solvability condition.

Remember that (4) is the same equation as (2), so the physical interpretation of (4) is the same as that of (2): namely that the total amount of heat flowing through the boundary \( C \) (per unit time) is 0.
In other words, for the temperature within $R$ to be at equilibrium, there must be no accumulation of loss of heat energy into or out of $R$.

Note: The same result can be deduced mathematically from the equation $\nabla^2 u = 0$ on $R$, using the divergence theorem. See equation (2.5.61) on page 85.

(b) Suppose $f(x)$ and $g(y)$ are constants, say $f(x) = B_1$ and $g(y) = B_2$. The condition of part (a) is that

$$\int_0^H g(y) \, dy + \int_0^L f(x) \, dx = 0, \quad \text{or} \quad \int_0^H B_2 \, dy + \int_0^L B_1 \, dx = 0,$$

or $B_2 H + B_1 L = 0$. Thus $\frac{-B_2}{L} = \frac{B_1}{H}$.

Now let $u(x,y) = A(x^2 - y^1)$. We see that $\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2A + (-2A) = 0$, so $u$ satisfies Laplace's equation. We also have to check the four boundary conditions: We have:

$$\frac{\partial u}{\partial x}(0,y) = 2Ax \bigg|_{x=0} = 0,$$

$$\frac{\partial u}{\partial y}(x,0) = -2Ay \bigg|_{y=0} = 0,$$

$$\frac{\partial u}{\partial x}(L,y) = 2Ax \bigg|_{x=L} = 2AL,$$
\[
\frac{du}{dy} (x, H) = -2A y \bigg|_{y=H} = -2AH.
\]

In order to satisfy the boundary conditions \( \frac{du}{dx} (L, y) = g(y) \) and \( \frac{du}{dy} (x, H) = f(x) \), we see from the last two equations that we have to choose the constant \( A \) so that both the equations \( 2AL = g(y) = B_2 \) and \( -2AH = f(x) = B_1 \) hold.

In other words, we must take \( A = \frac{B_2}{2L} \) and \( A = \frac{-B_1}{2H} \).

This would be impossible if \( \frac{B_2}{2L} \) and \( \frac{-B_1}{2H} \) were different constants, but as noted above, they are the same number because of the condition of part (a).

(c) Let \( \bar{bav} = \frac{1}{L} \int_0^L f(x) \, dx \) and \( \bar{gav} = \frac{1}{H} \int_0^H g(y) \, dy \).

Notice that \( \bar{bav} \) and \( \bar{gav} \) are constants. By part (b), the solution of the problem \( \nabla^2 u_1 = 0 \), with boundary conditions as shown in the diagram at left, will exit and will be given by
(4½) \( u_1 = A(x^2 - y^2) \)

\[ A = \left( \frac{B_2}{2L}, -\frac{B_1}{2H} \right) \]

(For consistency), provided the condition

(5) \( B_2 H + B_1 L = 0 \)

is satisfied, where here \( B_2 = g_{av} \) and \( B_1 = f_{av} \).

But we are given the condition \( 0 = \int_0^H g(y) \, dy + \int_0^L f(x) \, dx \),

which tells us that \( 0 = H \cdot \frac{1}{H} \int_0^H g(y) \, dy + L \cdot \frac{1}{L} \int_0^L f(x) \, dx \), or

\[ 0 = H \cdot g_{av} + L \cdot f_{av} \]

or \( 0 = B_2 H + B_1 L \). So (5) is satisfied, and the solution \( u_1 \) is valid.

Now let \( u_2 \) and \( u_3 \) be the solutions of \( \nabla^2 u_2 = 0 \) and \( \nabla^2 u_3 = 0 \) with the boundary conditions shown in the diagrams below:
in the problem for \( u_2 \).

Observe that because we have subtracted the constant \( b \) from \( f(x) \), it is now true that \( \int_0^L \frac{du_2}{dy}(x, H) \, dx = 0 \):

\[
\int_0^L \frac{du_2}{dy}(x, H) \, dx = \int_0^L [f(x) - b \cdot \text{av}] \, dx = \int_0^L f(x) \, dx - \int_0^L b \cdot \text{av} \, dx = \\
= \int_0^L f(x) \, dx - b \cdot \text{av} \cdot L = \\
= \int_0^L f(x) \, dx - \left( \frac{L}{L} \int_0^L f(x) \, dx \right) = \\
= \int_0^L f(x) \, dx - \int_0^L f(x) \, dx = 0.
\]

Therefore, from Problem 2.5.28 on page 85, we know that the problem for \( u_2 \) has a solution, and the solution is given by

\[
(6) \quad u_2(x, y) = A_0 + \sum_{n=1}^{\infty} A_n \cos\left(\frac{n \pi x}{L}\right) \cosh\left(\frac{n \pi y}{L}\right),
\]

where \( A_0 \) is arbitrary (not determined by the given conditions), and for \( n \geq 1 \),

\[
A_n = \frac{L}{n \pi} \cdot \frac{1}{\sinh\left(\frac{n \pi H}{L}\right)} \cdot \frac{2}{L} \int_0^L [f(w) - b \cdot \text{av}] \cos\left(\frac{n \pi w}{L}\right) \, dw.
\]

In fact, since \( \int_0^L b \cdot \text{av} \cos\left(\frac{n \pi w}{L}\right) \, dw = b \cdot \text{av} \int_0^L \cos\left(\frac{n \pi w}{L}\right) \, dw = 0 \) for \( n \geq 1 \), we can rewrite the formula for \( A_n \) as

\[
(7) \quad A_n = \frac{2}{n \pi \sinh\left(\frac{n \pi H}{L}\right)} \int_0^L f(w) \cos\left(\frac{n \pi w}{L}\right) \, dw.
\]
In the problem for \( u_3 \), the only difference from the problem for \( u_2 \) is that the variables \( x \) and \( y \) have been switched (with \( L \) replaced by \( H \) and \( b \) by \( g \)), so we get

\[
(8) \quad u_3(x, y) = B_0 + \sum_{n=1}^{\infty} B_n \cos\left(\frac{n\pi y}{H}\right) \coth\left(\frac{n\pi x}{H}\right),
\]

where \( B_0 \) is arbitrary and, for \( n \geq 1 \),

\[
(9) \quad B_n = \frac{2}{n\pi \sinh\left(\frac{n\pi L}{H}\right)} \int_0^H g(w) \cos\left(\frac{n\pi w}{H}\right) \, dw.
\]

Finally, we observe that the solution to the problem for \( u \) is given by \( u = u_1 + u_2 + u_3 \) (to see this, notice that

\[
\frac{\partial u}{\partial y}(x, y) = f(x) = b(x) + (b(x) + f(x)) + O
\]

\[
= \frac{\partial u_1}{\partial y}(x, y) + \frac{\partial u_2}{\partial y}(x, H) + \frac{\partial u_3}{\partial y}(x, H);
\]

and similarly for the other three boundary conditions).

Combining the formulas for \( u_1, u_2, u_3 \) in (4), (6), and (8), we get

\[
U = A(x^2 - y^2) + C_0 + \sum_{n=1}^{\infty} \left\{ A_n \cos\left(\frac{n\pi x}{L}\right) \coth\left(\frac{n\pi y}{L}\right) + B_n \cos\left(\frac{n\pi y}{H}\right) \coth\left(\frac{n\pi x}{H}\right) \right\}
\]

where

\[
A = \frac{gav}{2L} = \frac{-bav}{2L} = \frac{1}{2LH} \int_0^H g(w) \, dw = \frac{-1}{2HL} \int_0^L f(w) \, dw,
\]

\[
C_0 = A_0 + B_0 \text{ is an arbitrary constant, and } A_n \text{ and } B_n \text{ are given by the equations (7) and (9) for all } n \geq 1.
\]