Math 4513 Spring 2008
Assignment 4

(More problems taken from Ravi Vakil's excellent problem-solving website.)

1. The sequence $q_1, q_2, \ldots$ satisfies $q_n = 3q_{n-2} - 2q_{n-3}$, and $q_0 = 0$, $q_1 = 3$, $q_2 = 11$. Find a general formula for q_n.

2. What is $\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}^n$? (Prove your answer.)

3. Find a recurrence relation satisfied by $f_n = 3^n + 4^{n+1}$.

4. A gambling student tosses a fair coin and scores one point for each head that turns up and two points for each tail. Prove that the probability of the student scoring exactly n points at some time in a sequence of n tosses is $(2 + (-1/2)^n)/3$. (Hint: Let P_n denote the probability of scoring exactly n points at some time. Express P_n in terms of P_{n-1}, or in terms of P_{n-1} and P_{n-2}. Use this recurrence relation to given an inductive proof. Even better hint: you've been given the answer, so reverse-engineer the recurrence relation, and then try to prove it.)

5. Let F_n stand for the nth Fibonacci number, with the subscripts chosen so that we start with $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, $F_4 = 3$, and so on. Use induction to show that for every n, both the equations

$$F_n^2 + F_{n+1}^2 = F_{2n+1}$$

and

$$F_{n+1}^2 + 2F_nF_{n+1} = F_{2n+2}$$

hold.

(We had earlier the problem of proving just the first of these two equations. I think you'll find the problem of proving both of them to be much easier than the problem of proving just the first one!)