3. Define \( h = g - f \), then \( h \) is differentiable on \( R \).

Let \( x > 0 \) be given. By the Mean Value Theorem, there exists \( c \in (0, x) \) such that \( h(x) - h(0) = h'(c)(x - 0) \).

But \( h(0) = g(0) - f(0) = 0 - 0 = 0 \), and \( h'(c) = g'(c) - f'(c) \), so this implies \( h(x) = (g'(c) - f'(c)) \cdot x \).

Since \( g'(c) \geq f'(c) \), then \( g'(c) - f'(c) \geq 0 \), and \( x > 0 \), so \( h(x) \geq 0 \). Thus \( g(x) - f(x) \geq 0 \), so \( g(x) \geq f(x) \).

4. Let \( f(x) = 3x \) and \( g(x) = e^x + e^{2x} - 2 \). Then \( f(0) = 0 \) and \( g(0) = e^0 + e^0 - 2 = 0 \).

Also \( f'(x) = 3 \), and \( g'(x) = e^x + 2e^{2x} \geq e^0 + 2e^0 = 3 \) (since \( x > 0 \), and \( \frac{dx}{dx}(e^x) = e^x > 0 \), \( e^x \) is increasing, so \( e^x > e^0 \) and \( e^{2x} > e^0 \)). Therefore, by problem 3, \( f(x) \leq g(x) \) for all \( x > 0 \).

5. Let \( x \in R \) be given. By Taylor's Theorem with \( n = 2 \), there exists \( c \) between \( x \) and \( 0 \) such that

\[
\frac{f(x) - f(0) - f'(0)(x - 0)}{2} \leq \frac{f''(c)(x - 0)^2}{2}.
\]

Since \( f(0) = f'(0) = 0 \), this implies \( f(x) = \frac{f''(c)}{2} \cdot x^2 \).

But \( f''(c) \geq 0 \) by assumption, and \( x^2 \geq 0 \) (for all \( x \in R \)), so \( f''(c) \cdot x^2 \geq 0 \), so \( f(x) \geq 0 \).

(See also next page) 6. For all \( x \in [a, b] \), we have \( -|f(x)| \leq f(x) \leq |f(x)| \).

Let \( g(x) = -|f(x)| \) and \( h(x) = |f(x)| \). Then \( g \) and \( h \) are in \( R [a, b] \).

This is a fact we haven't proved in class yet, so I should have stated it as an assumption) and \( g \leq f \leq h \) on \( [a, b] \), so \( \int_a^b g \leq \int_a^b f \leq \int_a^b h \).

Hence \( \int_a^b (-|f|) \leq \int_a^b f \leq \int_a^b |f| \), so \( -\int_a^b |f| \leq \int_a^b f \leq \int_a^b |f| \).

This implies that \( |\int_a^b f| \leq \int_a^b |f| \). (Remember \( -\text{PSQ} \iff \text{P|Q} \).)
(7) Take \([a, b] = [0, 1]\), and define \(f(x) = \begin{cases} 1 & \text{for } 0 \leq x \leq \frac{1}{2} \\ 2 & \text{for } \frac{1}{2} < x < 1 \end{cases}\). Then \(f\) is a step function, so by results proven in class, \(f \in R[0, 1]\) and \(S(f) = 1 \cdot (\frac{1}{2} - 0) + (-1) \cdot (1 - \frac{1}{2}) = \frac{1}{2} - \frac{1}{2} = 0\). However, it is not true that \(f(x) = 0\) for all \(x \in [0, 1]\). This shows that the statement is false.

(8) Let \(\varepsilon = 1\), and let \(\delta > 0\) be given. Choose \(n \in \mathbb{N}\) such that \(\frac{1}{n} < \delta\). Taking \(P = P_n\) and \(\delta = 2_n\) for this \(n\), we have that there exist tagged partitions \(P\) and \(\delta\) of \([a, b]\) such that \(||P|| < \frac{1}{n} < \delta\) and \(||\delta|| < \frac{1}{n}\delta\), and \(|S(f, P) - S(f, \delta)| \geq \varepsilon\). Since \(\delta > 0\) is arbitrary, this shows that the statement "for all \(\varepsilon > 0\), there exists \(\delta > 0\) s.t. if \(||P|| < \delta\) and \(||\delta|| < \delta\)
they \(|S(f, P) - S(f, \delta)| \leq \varepsilon\)"
is false. It follows from the Cauchy criterion for integrability that \(f \notin R[0, 1]\).

Alternate proof of (6): Let \(\varepsilon > 0\) be given. Choose \(s_1 > 0\) s.t.
if \(||P|| < s_1\) then \(|S(f, P) - S^b(f)| < \frac{\varepsilon}{2}\), and choose \(s_2 > 0\)
s.t. if \(||P|| < s_2\) then \(|S(f, P) - S^b(f)| < \frac{\varepsilon}{2}\). Let \(P\) be
any partition of \([a, b]\) s.t. \(||P|| < \min(s_1, s_2)\).

Then \(|S(f, P) - S^b(f)| < \frac{\varepsilon}{2} \Rightarrow S(f, P) \leq S^b(f) + \frac{\varepsilon}{2}\) (1)

and \(|S(f, P) - S^b(f)| \leq |S(f, P) - S(f, \hat{P})| + \frac{\varepsilon}{2} \Rightarrow\)

But \(|S(f, \hat{P})| \leq \frac{\sum}{\hat{P}} |f(\xi)| (\hat{P} - \hat{P}') \leq \frac{\varepsilon}{2} \Rightarrow S(f, \hat{P}) \leq S^b(f) + \frac{\varepsilon}{2}\) (2)

Combining (1), (2), (3) gives \(|S^b(f)| < S^b(f) + \varepsilon\). Since \(\varepsilon > 0\) is arbitrary, this proves \(|S^b(f)| \leq S^b(f)|\).