1. Show that the weak topology on $\mathbb{C}^n$ is the same as the norm topology.

2. Let $C[a,b]$ be the Banach space of continuous functions on the finite interval $[a,b]$ in $\mathbb{R}$, with the supremum norm. Show that $C[a,b]$ is separable.

3. Let $M[a,b]$ be the Banach space of all (real-valued) additive set functions $\mu$ defined on the $\sigma$-algebra of Borel subsets of the finite interval $[a,b]$ in $\mathbb{R}$, with norm given by

$$ ||\mu|| = |\mu|([a,b]) = \int_{[a,b]} d|\mu|, $$

where $|\mu|$ is the total variation of $\mu$. Suppose $\{\mu_n\}$ is a sequence in $M[a,b]$ such that $||\mu_n|| \leq 1$ for all $n \in \mathbb{N}$. Show that there exists a subsequence $\{\mu_{n_k}\}$ of $\{\mu_n\}$ and an additive set function $\mu_0 \in M[a,b]$, with $||\mu_0|| \leq 1$, such that

$$ \lim_{n \to \infty} \int_{[a,b]} f d\mu_n = \int_{[a,b]} f d\mu_0 $$

for every continuous function $f$ on $[a,b]$. (Note: a version of this statement is still true for complex-valued measures on $[a,b]$, once you’ve defined the total variation of a complex measure and the integral of a continuous function with respect to a complex measure.)

4. Let $X$ be a Banach space. Show that if $\{x_n\}$ is a sequence in $X$ that converges weakly, then $\{||x_n||_X\}$ is bounded. (Hint: apply the Uniform Boundedness Principle to the functionals $j(x_n)$ in $X^{**}$.)

5. (a) Show that the sequence $\{e_n\}$ converges weakly to zero in $\ell^2$. (Here $e_n$ denotes the sequence with 1 as the $n$th element and zeroes elsewhere.)

(b) Find a sequence $\{f_n\}$ in $C[0,1]$ such that $||f_n|| = 1$ for all $n \in \mathbb{N}$, and $\{f_n\}$ converges weakly to 0.

Extra. Here are a couple more problems which are not assigned, but which I recommend as good exercises. They are not hard.

Show that if $X$ is a reflexive Banach space, then for every bounded linear functional $F$ on $X$, there exists $x \in X$ such that $||x|| = 1$ and $F(x) = ||F||$.

Show that $C[0,1]$ with the supremum norm is not a reflexive Banach space.