Math 6473
Assignment 3

1. Show that the weak topology on $\mathbf{C}^n$ is the same as the norm topology.

2. Let $C[a,b]$ be the Banach space of continuous functions on the finite interval $[a,b]$ in $\mathbf{R}$, with the supremum norm. Show that $C[a,b]$ is separable.

3. Let $M[a,b]$ be the Banach space of all (real-valued) additive set functions μ defined on the σ-algebra of Borel subsets of the finite interval $[a,b]$ in $\mathbf{R}$, with norm given by

$$\|\mu\| = |\mu|([a,b]) = \int_{[a,b]} d|\mu|,$$

where $|\mu|$ is the total variation of μ. Suppose $\{\mu_n\}$ is a sequence in $M[a,b]$ such that $\|\mu_n\| \leq 1$ for all $n \in \mathbf{N}$. Show that there exists a subsequence $\{\mu_{n_k}\}$ of $\{\mu_n\}$ and an additive set function $\mu_0 \in M[a,b]$, with $\|\mu_0\| \leq 1$, such that

$$\lim_{n\to\infty} \int_{[a,b]} f \, d\mu_n = \int_{[a,b]} f \, d\mu_0$$

for every continuous function f on $[a,b]$. (Note: a version of this statement is still true for complex-valued measures on $[a,b]$, once you've defined the total variation of a complex measure and the integral of a continuous function with respect to a complex measure.)

4. Let X be a Banach space. Show that if $\{x_n\}$ is a sequence in X that converges weakly, then $\{\|x_n\|_X\}$ is bounded. (Hint: apply the Uniform Boundedness Principle to the functionals $j(x_n)$ in X^{**}.)

5.
(a) Show that the sequence $\{e_n\}$ converges weakly to zero in ℓ^2. (Here e_n denotes the sequence with 1 as the nth element and zeroes elsewhere.)
(b) Find a sequence $\{f_n\}$ in $C[0,1]$ such that $\|f_n\| = 1$ for all $n \in \mathbf{N}$, and $\{f_n\}$ converges weakly to 0.

Extra. Here are a couple more problems which are not assigned, but which I recommend as good exercises. They are not hard.

Show that if X is a reflexive Banach space, then for every bounded linear functional F on X, there exists $x \in X$ such that $\|x\| = 1$ and $F(x) = \|F\|$.

Show that $C[0,1]$ with the supremum norm is not a reflexive Banach space.