**Math 6473**
**Assignment 2**

**1.** Give an example of a bounded linear map $A$ from a normed space $X$ to the complex numbers $\mathbf{C}$ such that there does not exist any $x$ in $X$ satisfying $|Ax| = \|A\|\,\|x\|$.

**2.** Suppose $X$ is a normed space, $Y$ is a Banach space, and $Z$ is a dense subspace of $X$. Show that if $A : Z \to Y$ is bounded and linear, then there exists a unique bounded linear map $B : X \to Y$ such that $Bx = Ax$ for all $x \in Z$.

**3.** Suppose $X$ is a normed space, and $A : X \to \mathbf{C}$ is linear. Define $N(A) = \{x \in X : Ax = 0\}$. Show that $A$ is bounded if and only if $N(A)$ is closed.

**4.** This exercise shows that completeness is an essential hypothesis of the uniform boundedness principle. Let $X$ be the space of all sequences $\{x_n\}$ in $\ell_\infty$ such that there exists $N \in \mathbf{N}$ so that $x_n = 0$ for all $n \geq N$.

(a) Show that $X$ is a subspace of $\ell_\infty$, but is not complete with respect to the $\ell_\infty$ norm.

(b) For each $k \in \mathbf{N}$, define the map $T_k : X \to \mathbf{C}$ by $T_k(\{x_n\}) = kx_k$. Show that the collection $\mathcal{F} = \{T_k\}_{k\in\mathbf{N}}$ is pointwise bounded, but is not uniformly bounded.

**5.** Let $X$ be as in problem **4** and define $T : X \to X$ by $(T(x_n))_n = (1/n)x_n$. Show that $T$ is bounded, one-to-one, and onto, but $T^{-1}$ is not bounded. Why does this not contradict the open mapping theorem?

**6.** Suppose $X$ and $Y$ are Banach spaces and $\{A_n\}$ is a sequence of bounded linear operators from $X$ to $Y$. We say $A_n$ *converges strongly* to the bounded operator $A : X \to Y$ if, for every $x \in X$, we have $A_n x \to Ax$.

(a) Prove that if $A_n$ converges in the norm of $B(X, Y)$ to $A$ (that is, $\|A_n - A\| \to 0$) then $A_n$ converges strongly to $A$.

(b) Give an example showing that a sequence of operators $A_n$ can converge strongly to an operator $A$, without converging in norm to $A$.

(c) Show that if $A_n$ converges strongly to $A$, then $A_n$ is uniformly bounded: that is, there exists $M < \infty$ such that $\|A_n\| \leq M$ for all $n$. (Hint: use the Uniform Boundedness Principle.)

**Extra.** This problem is not part of the assignment, it's just here in case you feel like doing it on your own. Its purpose is to show that, unlike finite-dimensional vector spaces, infinite-dimensional vector spaces never have unique Banach space topologies. It is taken from Robert E. Megginson's "Introduction to Banach Space Theory", and is a simplification and generalization of a construction we did in class for $\ell_1$.

(a) Suppose $(X, \|\cdot\|)$ is an infinite-dimensional normed space. Construct an unbounded one-to-one linear operator $T$ from $(X, \|\cdot\|)$ onto itself. (We showed how to construct an unbounded linear operator from $X$ to $\mathbf{C}$ in class, using a Hamel basis of $X$. A similar construction works here.)

(b) Let $\|x\|_T = \|Tx\|$ whenever $x \in X$. Show that $\|\cdot\|_T$ is a norm, that $T$ is an isometric isomorphism from $(X, \|\cdot\|_T)$ onto $(X, \|\cdot\|)$, and that $\|\cdot\|_T$ is a Banach norm if and only if $\|\cdot\|$ is a Banach norm. (An "isometric isomorphism" from $X$ to $Y$ is a map $T$ which is linear, one-to-one, and onto, and satisfies $\|Tx\|_Y = \|x\|_X$ for all $x \in X$. In this case, the fact that $T$ is an isometric isomorphism is obvious.)

(c) Show that the topologies induced by $\|\cdot\|$ and $\|\cdot\|_T$ are different.