Assignment 1

1. Show that in a metric space (X, d), if $x \in X$ and $r > 0$, then the closure $\overline{B_r(x)}$ of $B_r(x) = \{y : d(x, y) < r\}$ is contained in $\{y : d(x, y) \leq r\}$, but is not necessarily equal to $\{y : d(x, y) \leq r\}$.

2. Show that the topology defined on $\mathbf{R}$ by $d_1(x, y) = |x - y|$ is the same as that defined by $d_2(x, y) = |\tanh x - \tanh y|$, but $\mathbf{R}$ is not complete with respect to d_2.

3. Let c denote the space of all sequences $x = \{x_j\}_{j \in \mathbf{N}}$ of complex numbers such that $\lim_{j \to \infty} x_j$ exists. Show that c is a Banach space with norm given by $\|x\|_\infty := \sup_{j \in \mathbf{N}} |x_j|$.

4. Let c_0 denote the space of all sequences $x = \{x_j\}_{j \in \mathbf{N}}$ of complex numbers such that $\lim_{j \to \infty} x_j = 0$. Show that c_0 is a Banach space with norm $\|x\|_\infty$.

(Note: We show in class that if X is a Banach space and Y is a closed subspace of X, then Y is also a Banach space, with the norm inherited from X. Since we also show in class that ℓ^∞, the space of all bounded sequences, is a Banach space with norm $\|x\|_\infty$, then for problems 3 and 4, you only need to show that c and c_0 are closed subspaces of ℓ^∞.)