Math 5463 — Final Exam

1. Prove Hölder's inequality in the case when $1 \le p < \infty$. You may assume Young's inequality.

2. Prove that if the function f, defined on $\mathbf{R}^n$, is in L^1, then its Hardy-Littlewood maximal function f^* is in weak L^1.

3. Prove that if f is absolutely continuous on $[a, b]$, then f' exists a.e. in (a, b), and for all $x \in [a, b]$ we have

$$f(x) - f(a) = \int_a^x f'(t)\ dt.$$

4. Suppose $K(x, y)$ is defined for $(x, y) \in \mathbf{R}^2$, and $K \in L^2(\mathbf{R}^2)$. Suppose $f \in L^2(\mathbf{R})$, and define $g(x)$ for $x \in \mathbf{R}$ by

$$g(x) = \int_{-\infty}^{\infty} K(x, y) f(y)\ dy.$$

Prove that

$$\|g\|_{L^2(\mathbf{R})} \le \|K\|_{L^2(\mathbf{R}^2)} \|f\|_{L^2(\mathbf{R})}.$$

5. Prove that $f(x) = \sqrt{x}$ is absolutely continuous on $[0, 1]$.

6. Suppose $\{f_n\}$ is a sequence of functions in $L^p(\mathbf{R}^n)$ such that for all $g \in L^{p'}$,

$$\lim_{n \to \infty} \int_{\infty}^{\infty} f_n g\ dx = 0.$$

Prove that f_n converges in measure to the zero function in L^p.

7. Suppose $\{\phi_k\}$ is an orthonormal basis for L^2, and for $f \in L^2$ and $g \in L^2$, define $c_k = \langle f, \phi_k \rangle$ and $d_k = \langle g, \phi_k \rangle$. Show that

$$\langle f, g \rangle = \sum_k c_k \bar{d}_k.$$

(Hint: use Parseval's identity.)

8. Define the measure μ on the σ-algebra of Lebesgue measurable subsets of $\mathbf{R}$ by

$$\mu(E) = \begin{cases} 1 & \text{if } 0 \in E \\ 0 & \text{if } 0 \notin E. \end{cases}$$

a. Evaluate (with proof) the integral $g(x) = \int_{-\infty}^{\infty} 1\ d\mu$. (It has different values for different choices of x in $\mathbf{R}$.)

b. Answer the following questions, with proof.

(i) Is $g(x)$ of bounded variation on $[-1, 1]$?

(ii) Is $g(x)$ absolutely continuous on $[-1, 1]$?

(iii) Is $g(x)$ singular on $[-1, 1]$?