Math 5463
Test 2

1. *(25 points)* Show that if f is absolutely continuous and singular on $[a, b]$, then f is constant on $[a, b]$.

2. *(20 points)* Suppose $\{\phi_k\}$ is an orthonormal set of functions in L^2. For each $f \in L^2$ and $N \in \mathbf{N}$, define $s_N(f) = \sum_{k=1}^{N} \langle f, \phi_k \rangle \phi_k$.

a. Prove that $\|f - s_N(f)\|^2 = \|f\|^2 - \sum_{k=1}^{N} |\langle f, \phi_k \rangle|^2$.

b. Deduce Bessel's inequality from part **a**.

3. *(20 points)* Suppose H is a Hilbert space which contains an infinite orthonormal set $\{\phi_1, \phi_2, \phi_3, \dots\}$. Let $S = \{x \in H : \|x\| = 1\}$.

a. Show that if x and y are in S then $\|x - y\| = \sqrt{2}$.

b. Let $\mathcal{K}$ be the collection of all balls $B_{1/2}(x)$ of radius $1/2$ in H with centers x in S. Show that there is no finite subcollection of $\mathcal{K}$ which covers S.

4. *(15 points)* Let $f_N(x) = \sum_{k=1}^{N} \sin(kx)$ for $N \in \mathbf{N}$. Show that the sequence $\{f_N\}$ does not converge to a limit in $L^2[0, 2\pi]$.

5. *(20 points)*

a. Suppose f is monotone and absolutely continuous on $[a, b]$. Show that if $E \subset [a, b]$ and $|E| = 0$, then $|f(E)| = 0$.

b. Suppose f is the Cantor-Lebesgue function on $[0, 1]$. Find a set $E \subset [0, 1]$ such that $|E| = 0$ and $|f(E)| \neq 0$.