Math 5463
Test 1

1. *(20 points)* Prove Minkowski's inequality for L^p, in the case when $1 \le p < \infty$. You may use Hölder's inequality.

2. *(20 points)* Prove that L^p is complete, in the case when $1 \le p < \infty$.

3. *(20 points)* Suppose $1 \le p < \infty$. A function f on E is said to be in *weak* L^p on E if there exists a constant C such that for all $\alpha > 0$, the set $\{x \in E : |f(x)| > \alpha\}$ has measure less than $C/(\alpha^p)$.

a. Prove that if $f \in L^p(E)$ then f is in weak L^p on E.

b. Let $E = (0, \infty)$ and define $f(x) = |x|^{-1/p}$ for $x \in E$. Show that f is in weak L^p on E but is not in $L^p(E)$.

4. *(20 points)*

a. Suppose f is continuous with compact support in $\mathbf{R}$. Prove

$$\lim_{h\to\infty} \int_{-\infty}^{\infty} |f(xh)|\, dx = 0.$$

b. Prove that for all f in $L^1(\mathbf{R})$,

$$\lim_{h\to\infty} \int_{-\infty}^{\infty} |f(xh)|\, dx = 0.$$

(Hint: use part **a.**)

5. *(20 points)* Suppose $0 < r \le q < \infty$, and suppose $E \subset \mathbf{R}$, with $|E| < \infty$. Prove that for all $f \in L^q(E)$,

$$\|f\|_{L^r(E)} \le \|f\|_{L^q(E)} |E|^{\frac{1}{r} - \frac{1}{q}}.$$

(Hint: use Hölder's inequality.)