Solutions to Exam 2

(1) Since f and g are integrable on $\mathbb{R}$, then by a theorem from class, $\int_{\mathbb{R}} |f|\,dx < \infty$ and $\int_{\mathbb{R}} |g|\,dx < \infty$. Hence also $\int_{\mathbb{R}} (|f| + |g|)\,dx$ $= \int_{\mathbb{R}} |f|\,dx + \int_{\mathbb{R}} |g|\,dx < \infty$. Now $|\max(f,g)|$ $\leq \max(|f|, |g|) \leq |f| + |g|$, so by monotonicity of integration, $\int_{\mathbb{R}} |\max(f,g)| \leq \int_{\mathbb{R}} (|f| + |g|)\,dx < \infty$.

Therefore $|\max(f,g)|$ and hence also $\max(f,g)$ are integrable.

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(2) (a) For all $x > 0$, $|e^{-nx}| = e^{-nx} < e^0 = 1$, so $|e^{-nx} f(x)| \leq |f(x)|$. Since f is integrable on $[0,\infty)$, so is $|f|$. Therefore we can apply the Dominated Convergence Theorem to the sequence $f_n(x) = e^{-nx} f(x)$, using $g = |f|$ as the dominating function. There results

$$\lim_{n\to\infty} \int_0^\infty e^{-nx} f(x)\,dx = \int_0^\infty \lim_{n\to\infty} (e^{-nx} f(x))\,dx = \int_0^\infty 0\,dx = 0,$$

because $\lim_{n\to\infty} e^{-nx} f(x) = 0$ for every $x > 0$.

(2)(b) Let $f_n(x) = \chi_{[n,\infty)}(x) \cdot f(x)$. Then for all $n \in \mathbb{N}$, $|f_n| \leq |f|$, so we can apply the Dominated Convergence Theorem as in part (a).

Since $\lim_{n\to\infty} \chi_{(n,\infty)}(x) \cdot f(x) = 0 \cdot f(x) = 0$ for all $x \in [0,\infty)$, this gives $\lim_{n\to\infty} \int_n^\infty f(x)\,dx = \lim_{n\to\infty} \int_{[0,\infty)} f_n\,dx$

$$= \int_{[0,\infty)} \lim_{n\to\infty} f_n(x)\,dx = \int_{[0,\infty)} 0\,dx = 0.$$

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(3) Let $g(x) = (f(x))^2$. Then $\{x \in E : f > \alpha\} = \{x \in E : g > \alpha^2\}$. By Chebyshev's Inequality,

$$m(\{g > \alpha^2\}) \leq \frac{1}{\alpha^2} \int_E g\,dx, \text{ so}$$

$$m(\{f > \alpha\}) \leq \frac{1}{\alpha^2} \int_E f^2\,dx.$$

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(4) Let $g(x) = \sum_{n=1}^{\infty} \frac{f_n(x)}{2^n}$. From a Theorem in class, we know $\int_E g\,dx = \sum_{n=1}^{\infty} \int_E \frac{f_n(x)}{2^n}\,dx$.

But the right-hand side is less than $\sum_{n=1}^{\infty} \frac{1}{2^n} \cdot 1$, so it is finite. Since $\int_E g\,dx < \infty$, another Theorem from class tells us that $g(x) < \infty$ a.e. on E.

5. Let g be a bounded measurable function on $[0,1]$, and let $\varepsilon > 0$ be given. By a Theorem from class, there exists a simple function h on $[0,1]$ such that

$$|g(x) - h(x)| < \frac{\varepsilon}{2M} \quad \text{for all } x \in [0,1].$$

Also, since $\lim_{n\to\infty} \int_{[0,1]} h\, b_n\, dx = 0$, there exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\left| \int_{[0,1]} h\, b_n\, dx \right| < \frac{\varepsilon}{2}.$$

Therefore, for all $n \geq N$,

$$\left| \int_0^1 g\, b_n\, dx \right| = \left| \int_{[0,1]} (g-h)\, b_n\, dx + \int_{[0,1]} h\, b_n\, dx \right|$$

$$\leq \int_{[0,1]} |g-h|\, |b_n|\, dx + \left| \int_{[0,1]} h\, b_n\, dx \right|$$

$$< \int_{[0,1]} \frac{\varepsilon}{2M} \cdot M\, dx + \frac{\varepsilon}{2}$$

$$= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

This proves $\lim_{n\to\infty} \int_{[0,1]} g\, b_n\, dx = 0$.