Math 5453 — Fall 2012
Final Exam

(For these problems, you may cite any result that we have seen in class, without having to prove it.)

1. Suppose $E \subseteq \mathbf{R}$ is measurable, with $m(E) < \infty$. Prove that for every $\epsilon > 0$, there exists a set H which is a finite union of open intervals such that

$$m\left((E \sim H) \cup (H \sim E)\right) < \epsilon.$$

2. Suppose the E_n, $n = 1, 2, 3 \ldots$, and E are measurable sets which are all contained in a measurable set F of finite measure. Suppose χ_{E_n} converges to χ_E pointwise on $\mathbf{R}$. Prove that $m(E_n) \to m(E)$. (Hint: use the dominated convergence theorem.)

3. Give an example of a measurable set E for which the statement

$$m(E) = \sup\{m(U) : U \text{ is open and } U \subseteq E\}$$

is false.

4. Suppose f is integrable on $\mathbf{R}$. Show that for every ϵ, there exists $K > 0$ such that

$$\int_{\mathbf{R} \sim [-K,K]} |f| < \epsilon.$$

5. Prove that if f_n is a sequence of non-negative measurable functions on a measurable set E such that

$$\lim_{n \to \infty} \int_E f_n = 0,$$

then f_n converges in measure to 0.

6. Prove that if f and g are functions of bounded variation on $[a, b]$, then $f + g$ is also of bounded variation.

7. Suppose f is continuous on $\mathbf{R}$. Prove that

$$\lim_{n \to \infty} \int_0^1 \left(f(x + \frac{1}{n}) - f(x)\right)\, dx = 0.$$

8. Prove that the Cantor-Lebesgue function is not absolutely continuous on [0,1].

9. Suppose f is monotone increasing and absolutely continuous on $\mathbf{R}$, and suppose E is a subset of $\mathbf{R}$ with measure zero. Prove that $f(E)$ has measure zero.

10. Show that if ϕ is convex on $[a, b]$, then there exists a function g on $[a, b]$ such that $g = \phi'$ a.e. in $[a, b]$ and g is differentiable a.e. in $[a, b]$.