

## Problems from Rudin

A student asked for some extra problems beyond the ones in the book. As a small sample, here are some problems from Rudin's "Real and Complex Analysis". I don't promise that these are representative of problems on our test. Several of these have already been done in our class, some might be too easy, or too hard to put on a test. But I think they all provide some way to check your understanding of the concepts.

1. Suppose  $f_n$  is a sequence of non-negative measurable functions on  $E$  and  $f_1 \geq f_2 \geq f_3 \geq \dots$  on  $E$ . Suppose that  $f_1$  is integrable on  $E$ . Prove that

$$\lim_{n \rightarrow \infty} \int_E f_n \, dx = \int_E f \, dx.$$

Show also that this conclusion does *not* follow if the assumption that  $f_1$  is integrable is omitted.

2. Let  $\Sigma$  be the collection of all sets  $E \in \mathbf{R}$  such that either  $E$  or  $\mathbf{R} - E$  is at most countable, and define  $\mu(E) = 0$  if  $E$  is countable and  $\mu(E) = 1$  if  $\mathbf{R} - E$  is countable. Prove that  $\Sigma$  is a  $\sigma$ -algebra and, for any countable disjoint collection  $\{E_k\}$  of sets in  $\Sigma$ ,

$$\mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mu(E_k).$$

3. Does there exist an infinite  $\sigma$ -algebra which has only countably many members?

4. Suppose  $m(E) < \infty$  and  $f_n$  is a sequence of bounded measurable functions which converges uniformly to  $f$  on  $E$ . Show that

$$\lim_{n \rightarrow \infty} \int_E f_n \, dx = \int_E f \, dx.$$

Also show that this conclusion is not true if we omit the hypothesis that  $m(E) < \infty$ .

5. If  $f$  is a Lebesgue measurable function on  $\mathbf{R}$ , show there is a Borel measurable function  $g$  on  $\mathbf{R}$  such that  $f = g$  a.e. on  $\mathbf{R}$ . (A function  $g$  is Borel measurable if for every  $c \in \mathbf{R}$ ,  $\{x : g(x) > c\}$  is a set in the Borel  $\sigma$ -algebra.)

6. Construct a sequence of continuous functions  $f_n$  on  $[0, 1]$  such that  $0 \leq f_n \leq 1$  and such that

$$\lim_{n \rightarrow \infty} \int_{[0,1]} f_n \, dx = 0,$$

but such that the sequence  $\{f_n(x)\}$  converges for no  $x \in [0, 1]$ .

7. It is easy to guess the values of

$$\lim_{n \rightarrow \infty} \int_0^n \left(1 - \frac{x}{n}\right)^n e^{x/2} \, dx$$

and

$$\lim_{n \rightarrow \infty} \int_0^n \left(1 + \frac{x}{n}\right)^n e^{-2x} \, dx$$

Prove that your guesses are correct.

8. For every Lebesgue measurable set  $E$  in  $\mathbf{R}$ , prove that  $m(-E) = m(E)$ , where  $-E = \{-x : x \in E\}$ . Then use that to prove that for every integrable function  $f$  on  $\mathbf{R}$ ,

$$\int_{\mathbf{R}} f(x) \, dx = \int_{\mathbf{R}} f(-x) \, dx.$$

### Problems from web sites for other real analysis courses

As I've mentioned before, you can pick up some useful bits of knowledge by browsing the websites for graduate real analysis classes at other universities. One good example is Prof. Terence Tao's Math 245a website at U.C. San Diego, which contains a blog related to the class with a number of exercises:

<http://terrytao.wordpress.com/category/teaching/245a-real-analysis/>

One particularly interesting looking entry gives a long list of strategies for solving real analysis problems:

<http://terrytao.wordpress.com/2010/10/21/245a-problem-solving-strategies/>

This blog was eventually turned into a book, and a .pdf copy of a preliminary version of the book is at <http://terrytao.files.wordpress.com/2011/01/measure-book1.pdf>

(The finished book, "An introduction to measure theory" by Terence Tao, is available from the American Mathematical Society.) The exercises in the blog and a number of others are included in the book, scattered throughout the text.