ANSWERS TO QUESTIONS ON EXAM 2

(1) (a) $\sin(\log x) = 0 \iff \log x = n\pi$ for $n \in \mathbb{Z}$ (2)

$\iff x = e^{n\pi}$ for some $n \in \mathbb{Z}$ (2)

(b) The sequence $x_1 = e^{-\pi}, x_2 = e^{-2\pi}, x_3 = e^{-3\pi}, \dots$ consists of points where $\sin(\log x) = 0$, and the sequence has (2) an accumulation point at 0. If $f(z)$ were an entire function such that $f(x) = \sin(\log x)$ for $x > 0$, then f would have zeroes which accumulate at zero, which is impossible (4) since f is not constant.

---

(2) Since $f(z)$ is entire, so is $e^{f(z)}$. But $|e^{f(z)}| =$

$= |e^{u+iv}| = |e^u||e^{iv}| \underset{(2)}{=} e^u \le e^{|u|} \le e^M$ for all $z \in \mathbb{C}$.

So $e^{f(z)}$ is a bounded entire function (2), and is therefore constant by Liouville's Theorem (2). Since $e^{f(z)} = e^{f(0)}$ for all $z \in \mathbb{C}$, then for all $z \in \mathbb{C}$ there (2) exists $n = n(z) \in \mathbb{N}$ such that $f(z) = f(0) + 2\pi i\, n(z)$. Since $f(z)$ is continuous, so is $n(z)$. But the only integer-valued continuous functions are constant (2) functions, so $n(z)$ is constant. Hence $f(z)$ is constant.

---

(3) The problem was incorrect as stated, as is shown by the example $f(z) = \dfrac{z^2}{1-z}$ on $D(0,1)$. A corrected version is: Suppose f is a holomorphic function on an open set $\mathcal{U}$ containing $\overline{D(P,r)}$, and $f(P) = f'(P) = 0$. Show there exists $M > 0$ such that for all $z \in D(P,r)$, $|f(z)| \le M|z-P|^2$.

To prove this, observe that the assumption on f implies that there exists $r_1 > r$ such that $D(P, r_1) \subseteq \mathcal{U}$ and f is holomorphic on $D(P, r_1)$. The Taylor series of $f(z)$ converges to f on $\overline{D(P,r)}$, and since $f(P) = f'(P) = 0$ then $f(z) = \sum_{n=2}^{\infty} \left(\frac{f^{(n)}(P)}{n!}\right)(z-P)^n = (z-P)^2 g(z)$, where $g(z)$ is given by a convergent power series on $D(P, r_1)$ and is therefore continuous, and hence bounded, on $\overline{D(P,r)}$. So there exists $M > 0$ so that $|g(z)| \le M$ for all $z \in D(P,r)$, and thus $|f(z)| = |z-P|^2 |g(z)| \le |z-P|^2 M$.

---

4 (a) For $0<|z|<1$, (2) $f(z) = \frac{1}{z}\frac{1}{(1+z^2)} = \frac{1}{z}\left(1 - z^2 + (z^2)^2 - (z^2)^3 + \cdots\right)$

(3) $= \frac{1}{z} - z + z^3 - z^5 + z^7 + \cdots$

(b) For $|z|>1$, (2) $f(z) = \frac{1/z^2}{z\left(\frac{1}{z^2}+1\right)} = \frac{1}{z^3}\left(1 - \frac{1}{z^2} + \left(\frac{1}{z^2}\right)^2 - \left(\frac{1}{z^2}\right)^3 + \cdots\right)$

(3) $= \frac{1}{z^3} - \frac{1}{z^5} + \frac{1}{z^7} - \frac{1}{z^9} + \cdots$

---

5 The poles of $f(z) = \frac{(3z+2)^2}{z(z-1)(2z+5)}$ within $|z|=2$ are (2) at $z=0$ and $z=1$, so by the Residue Theorem,

(2)
$$\int_{|z|=2} f(z)\,dz = 2\pi i\left(\mathrm{Res}_f(0) + \mathrm{Res}_f(1)\right) = 2\pi i\left(\frac{2^2}{(-1)(5)} + \frac{5^2}{(1)(7)}\right)$$
(2) (2) (2)

$$= 2\pi i\left[-\frac{4}{5} + \frac{25}{7}\right] = 2\pi i\left(\frac{97}{35}\right) = \frac{194\pi i}{35}.$$

---

6 (a) Let $h(\zeta) = \frac{1}{(\zeta-P)(\zeta-z)}$; Then h has poles within (3) C at P and at z, so by the Residue Theorem

$$\int_C h(\zeta)\,d\zeta = 2\pi i\left[\mathrm{Res}_h(P) + \mathrm{Res}_h(z)\right] = 2\pi i\left[\frac{1}{P-z} + \frac{1}{z-P}\right] = 0.$$

(b) Since f is holomorphic on $\bar{U}$ then f is differentiable at P, so $\lim\limits_{z \to P} \frac{f(z)-f(P)}{z-P} = f'(P)$

3 exists. Since $\lim\limits_{z \to P} g(z)$ exists, then g must be bounded in a neighborhood of P. Therefore g has a removable singularity at P, and can thus be extended to a holomorphic function on $\bar{U}$.

(c.) Since g can be extended to a holomorphic function $\hat{g}$ on $\bar{U}$, and C and the ~~the~~ interior of C are contained within $\bar{U}$, then by the Cauchy integral formula, for all $z \in D(P,r)$,

Note: It's perhaps easier to do part (c) by simply applying the residue theorem to $g(z)$.

2 $$\hat{g}(z) = \frac{1}{2\pi i} \int_C \frac{\hat{g}(s)}{s-z}\, ds .$$

But for $s \in C$, we have $s \neq P$ so $\hat{g}(s) = g(s) = \frac{f(s)-f(P)}{(s-P)}$. Thus for $z \in D(P,r) \setminus \{P\}$,

$$g(z) = \frac{1}{2\pi i} \int_C \frac{f(s)-f(P)}{(s-P)(s-z)}\, ds .$$

$$\Rightarrow \quad g(z) = \frac{1}{2\pi i} \int_C \frac{f(s)}{(s-P)(s-z)}\, ds - \frac{1}{2\pi i} f(P) \int_C \frac{1}{(s-P)(s-z)}\, ds$$

2 $$\Rightarrow \quad g(z) = \frac{1}{2\pi i} \int_C \frac{f(s)}{(s-P)(s-z)}\, ds - 0, \quad \text{(by (a))}$$

as desired.