Complex Analysis I
Exam 2

1. Define $f : (0, \infty) \to \mathbf{R}$ by $f(x) = \sin(\log x)$.
a. Find the values of $x \in (0, \infty)$ at which $f(x) = 0$.
b. Use the result of part **a** to answer the question: can f be extended to an entire function on the complex plane?

2. Suppose $f = u + iv$ is entire and there exists $M > 0$ such that $|u(z)| \le M$ for all $z \in \mathbf{C}$. Show that f is constant. Hint: apply Liouville's theorem to $\exp(f(z))$.

3. Suppose f is a holomorphic function on $D(P, r)$, and $f(P) = f'(P) = 0$. Show that there exists $M > 0$ such that for all $z \in D(P, r)$,

$$|f(z)| \le M|z - P|^2.$$

4. Find two Laurent series for $f(z) = \dfrac{1}{z(1 + z^2)}$ in powers of z, and specify where each one is valid.

5. Evaluate the integral $\displaystyle\int_C \frac{(3z + 2)^2}{z(z - 1)(2z + 5)}\, dz$, where C is the circle $\{|z| = 2\}$, traversed counterclockwise.

6. Suppose $U \subseteq \mathbf{C}$ is open, and $\overline{D}(P, r) \subseteq U$. Let C be the circle $\{|z - P| = r\}$, with positive orientation.
a. Show that if $z \in D(P, r) \backslash \{P\}$, then

$$\int_C \frac{1}{(\zeta - P)(\zeta - z)}\, d\zeta = 0.$$

b. Show that if $f(z)$ is a holomorphic function on U, then the function $g(z) = \dfrac{f(z) - f(P)}{z - P}$ is also holomorphic on U.

c. Show that if f and g are as in part **b**, then for all $z \in D(P, r) \backslash \{P\}$,

$$g(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - P)(\zeta - z)}\, d\zeta.$$