Math 5403 — Calculus of Variations
Assignment 3

1. Find the solution $y(x)$, $z(x)$ of the Euler-Lagrange equation for the functional

$$J[y,z] = \int_0^{\pi/2} (y'^2 + z'^2 + 2yz)\ dx$$

subject to the boundary conditions

$$y(0) = 0, \quad y(\pi/2) = 1, \quad z(0) = 0, \quad z(\pi/2) = 1.$$

(Note: you can use the two Euler-Lagrange equations to obtain a single linear fourth-order equation for y, which you can solve by means of the characteristic equation. If you're rusty on that solution method, check the Wikipedia page for "Characteristic equation (calculus)".)

2. Find the solution of the Euler-Lagrange equation for the functional

$$J[y,z] = \int_0^1 (1 + y''^2)\ dx$$

subject to the boundary conditions

$$y(0) = 0, \quad y'(0) = 1, \quad y(1) = 0, \quad y'(1) = 1.$$

3. Which curve minimizes the integral

$$\int_0^1 \left(\frac{y'^2}{2} + yy' + y' + y \right)\ dx$$

when the values of y are not specified at the endpoints?
Answer. $y = \frac{1}{2}(x^2 - 3x + 1)$.

4. Suppose y is a solution of the Euler-Lagrange equation for the functional

$$J[y] = \int_a^b F(x, y, y', y'')\ dx.$$

a) Show that if F does not depend on y, then

$$F_{y'} - \frac{d}{dx} F_{y''} = C$$

for all $x \in [a,b]$, where C is a constant.

b) Show that if F does not depend on x, then

$$F - y'F_{y'} + y' \frac{d}{dx} F_{y''} - y''F_{y''} = C$$

for all $x \in [a,b]$, where C is a constant.

5. Suppose $a > 0$ and $L > 2a$, and consider the problem of finding the minimizer $y(x)$ for the functional

$$J[y] = \int_{-a}^{a} y\sqrt{1 + y'^2}\ dx$$

subject to the constraints $y(-a) = 0$, $y(a) = 0$, and

$$K[y] = \int_{-a}^{a} \sqrt{1 + y'^2}\; dx = L.$$

(This problem has a physical interpretation: $J[y]$ gives the center of gravity of a hanging cable whose position is given by the function $y(x)$, and $K[y]$ is the length of the cable. Since a hanging cable will be in equilibrium when its center of gravity is lowest, then the minimizer will give the equilibrium position of a hanging cable. For this reason it is called a *catenary*, from the Latin word for "chain".)

a) Find the general solution of the Euler-Lagrange equation. Your answer will involve a Lagrange multiplier λ and two arbitrary constants. (Note: the integral involved is easiest done by substitution, using the formula $\dfrac{d}{dx} \cosh^{-1} u = \dfrac{1}{\sqrt{u^2 - 1}}$.)

b) Use the constraints to determine the value of λ and the constants. You will not be able to give these as explicit functions of a and L, but you should be able to find fairly simple equations giving the constants implicitly in terms of a and L.