ANSWERS TO EXAM 2

(1) (a) First we show (x_n) is bounded below by 2 and (2)
[16] bounded above by 10, using induction.

Since $x_1 = 2$, then $2 \le x_1 \le 10$. (2)

Assume $2 \le x_n \le 10$. Then $1 \le x_n - 1 \le 9$, so (4)
$\sqrt{1} \le \sqrt{x_n - 1} \le \sqrt{9}$, so $1 \le \sqrt{x_n - 1} \le 3$, so $4 \le \sqrt{x_n - 1} + 3 \le 6$,
so $4 \le x_{n+1} \le 6$, so $2 \le x_{n+1} \le 10$. This proves the statement by induction.

Next we use induction to show (x_n) is increasing. (2)

Since $x_1 = 2$ and $x_2 = \sqrt{2-1} + 3 = 4$, then $x_1 \le x_2$. (2)

Assume $x_n \le x_{n+1}$. Then $x_n - 1 \le x_{n+1} - 1$, and both $x_n - 1 \ge 0$ and $x_{n+1} - 1 \ge 0$ (since $x_n \ge 2$ and $x_{n+1} \ge 2$ as shown above), so $\sqrt{x_n - 1} \le \sqrt{x_{n+1} - 1}$, so $\sqrt{x_n - 1} + 3 \le \sqrt{x_{n+1} - 1} + 3$, so $x_{n+1} \le x_{n+2}$, as desired. (4)

(b) ~~Since (x_{n+1}) is a subsequence of (x_n) then both x_{n+1}~~
[9] By the monotone convergence Theorem, since (x_n) is increasing and bounded, then (x_n) converges to a limit, which we will call x. (3)

Since (x_{n+1}) is a subsequence of (x_n), then by a Theorem from class, (x_{n+1}) converges to the same limit x.

Since $x_{n+1} = \sqrt{x_n - 1} + 3$ for all $n \in \mathbb{N}$, then (2)
$\lim (x_{n+1}) = \lim (\sqrt{x_n - 1} + 3)$, so $x = \lim (\sqrt{x_n - 1} + 3)$.

From Theorem 3.2.3, $\lim (x_n - 1) = x - 1$, and since $x_n \ge 2$ for all n, then ~~[illegible]~~ $x_n - 1 \ge 0$ for all n, so by a Theorem from the text, $\lim \sqrt{x_n - 1} = \sqrt{x - 1}$.

Then by Theorem 3.2.3 again, $\lim (\sqrt{x_n - 1} + 3) = \sqrt{x - 1} + 3$.

(cont'd)

So $x = \sqrt{x-1} + 3$. Therefore $x - 3 = \sqrt{x-1}$, so

$(x-3)^2 = x-1$, so $x^2 - 6x + 9 = x - 1$, so $x^2 - 7x + 10 = 0$,

so $(x-2)(x-5) = 0$, so $x = 2$ or $x = 5$. (2)

Since $x_2 = 4$ and (x_n) is increasing, then $x_n \geq 4$ for all $n \in \mathbb{N}$ such that $n \geq 2$. By a Theorem from class it follows that $x \geq 4$.

(2)

So $x = 5$.

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(2) We will prove the statement by showing that if $(1/x_n)$ does converge, then (x_n) cannot converge to 0.

(2)

If $(1/x_n)$ converges, then it is bounded. So there exists $M \in \mathbb{R}$, with $M > 0$, such that $|1/x_n| \leq M$ for all $n \in \mathbb{N}$. (4) So $\frac{1}{M} \leq |x_n|$ for all $n \in \mathbb{N}$. (2)

Since $M > 0$ then $\frac{1}{M} > 0$. Therefore, if we let $\varepsilon = \frac{1}{M}$, then assuming $\lim (x_n) = 0$, we would get that there exists $K \in \mathbb{N}$ such that for all $n \geq K$,

$$|x_n - 0| < \frac{1}{M}.$$ (4)

But this contradicts the fact that $|x_n| \geq \frac{1}{M}$ for all $n \in \mathbb{N}$. So (x_n) cannot converge to 0. (3)

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(3) If (x_n) ~~has a subsequence~~ converges to some limit x, then all its subsequences must also converge to x. Now suppose (x_n) has one

(5)

subsequence, all of whose terms are greater than or equal to 2. Then ~~of course~~ by a Theorem from class, since this subsequence converges to x, then $x \geq 2$. 5 Similarly if (x_n) has a subsequence whose terms are all less than or equal to 1, then $x \leq 1$. 5 We can't have both $x \geq 2$ and $x \leq 1$, so (x_n) cannot converge.

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4 As shown in class, 2 $\lim(1+\frac{1}{n})^n = e$. Since $(1+\frac{1}{3n})^{3n}$ is a subsequence of $(1+\frac{1}{n})^n$, then $\lim(1+\frac{1}{3n})^{3n} = e$. 5 Now $(1+\frac{1}{3n})^n = \left[(1+\frac{1}{3n})^{3n}\right]^{\frac{1}{3}}$, so by a Theorem from class, $\lim(1+\frac{1}{3n})^n = \left(\lim\left[(1+\frac{1}{3n})^{3n}\right]\right)^{\frac{1}{3}} = e^{\frac{1}{3}}$. 3

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5 (a) [10] For all $n \geq 3$ we have $n^n \geq 3^n$, so $\frac{2^n}{n^n} \leq \frac{2^n}{3^n}$, so $\left(\frac{2^n}{n^n}\right) \leq \left(\frac{2}{3}\right)^n$. 3 We know that $\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n$ converges, since it is a geometric series with $r = \frac{2}{3}$, and $|r| = \frac{2}{3}$ is less than 1. 3 So also $\sum_{n=3}^{\infty}\left(\frac{2}{3}\right)^n$ converges.

Since $0 \leq \frac{2^n}{n^n} \leq \left(\frac{2}{3}\right)^n$ for $n \geq 3$, it follows from the comparison test that $\sum_{n=3}^{\infty}\frac{2^n}{n^n}$ converges. 3 Since $\sum_{n=3}^{\infty}\frac{2^n}{n^n}$ and $\sum_{n=1}^{\infty}\frac{2^n}{n^n}$

have partial sums which differ by a constant,

Then $\sum_{n=1}^{\infty} \frac{2^n}{n^n}$ must also converge. ①

[10] (b) We know that if $\sum_{n=1}^{\infty} a_n$ converges then $\lim a_n = 0$. ③

Therefore, since $\lim \left(\frac{n+1}{n+2}\right) = \lim \left(\frac{1+\frac{1}{n}}{1+\frac{2}{n}}\right) = \frac{1+0}{1+0} = 1$, ④

then $\sum_{n=1}^{\infty} \frac{n+1}{n+2}$ cannot converge.
③

(6.) Let $L = \lim_{x \to 0} f(x)$. We will use the sequential criterion for limits to prove that $\lim_{x \to 2} g(x) = L$.

Suppose (x_n) is a sequence such that (i) $x_n \in \mathbb{R}$ for all $n \in \mathbb{N}$, (ii) $x_n \neq 2$ for all $n \in \mathbb{N}$, and (iii) $\lim (x_n) = 2$. ④

Define $y_n = x_n - 2$ for all $n \in \mathbb{N}$. Then (i) $y_n \in \mathbb{R}$ for all $n \in \mathbb{N}$, (ii) $y_n \neq 0$ for all $n \in \mathbb{N}$, and (iii) $\lim (y_n) = \lim (x_n - 2) = 2 - 2 = 0$ (by Th 3.2.3). ④

Since $\lim_{x \to 0} f(x) = L$, it then follows from the sequential criterion for limits that $\lim f(y_n) = L$. ③

But $g(x_n) = f(x_n - 2) = f(y_n)$. So $\lim g(x_n) = L$.

Now it follows from the reverse direction of the sequential criterion that $\lim_{x \to 2} g(x) = L$. ④

Note: It's also possible to do this problem using the ε-δ definition of limit. See next page ⟶

(6.) (alternate solution)

Let $L = \lim_{x \to 0} f(x)$. We will use the ε-δ definition of limit to prove that $\lim_{x \to 2} g(x) = L$.

Let $\varepsilon > 0$ be given.

By assumption, there exists $\delta > 0$ such that if $0 < |x - 0| < \delta$ then $|f(x) - L| < \varepsilon$.

Now suppose $0 < |x-2| < \delta$. Let $y = x-2$.

Then $|y-0| = |x-2|$, so $0 < |y-0| < \delta$. So, by the above, we know $|f(y) - L| < \varepsilon$. Hence $|f(x-2) - L| < \varepsilon$. So $|g(x) - L| < \varepsilon$.

We have shown there exists $\delta > 0$ such that if $0 < |x-2| < \delta$ then $|g(x) - L| < \varepsilon$. So $\lim_{x \to 2} g(x) = L$.