olution to example problem from class : (cf. problem 4.1.14 in text).

Let $f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$

Show that

(a) f is continuous at $c = 0$

(b) f is not differentiable at $c = 0$

(c) f is not continuous at c if $c \neq 0$.

(a.) Let $\varepsilon > 0$ be given.

Define $\delta = \varepsilon$.

Then $\delta > 0$.

Let $x \in \mathbb{R}$ be given such that $|x - 0| < \delta$

Then $|x - 0| < \varepsilon$.

There are two possibilities: either (i) $x \in \mathbb{Q}$ or (ii) $x \notin \mathbb{Q}$

(i) If $x \in \mathbb{Q}$, then $f(x) = x$ (given)

So $|f(x) - 0| = |x - 0|$

So $|f(x) - 0| < \varepsilon$.

(ii) If $x \notin \mathbb{Q}$, then $f(x) = 0$ (given)

So $|f(x) - 0| = |0 - 0| = 0$

So $|f(x) - 0| < \varepsilon$

In either case, $|f(x) - 0| < \varepsilon$.

But $f(0) = 0$ (given: $0 \in \mathbb{Q}$, so $f(0) = 0$).

So $|f(x) - f(0)| < \varepsilon$.

This proves that if $|x - 0| < \delta$, then $|f(x) - f(0)| < \varepsilon$.

This proves f is cont. at $c = 0$. $\boxed{QED}$

(b.) We have to prove that $\lim\limits_{x \to c} \dfrac{f(x) - f(c)}{x - c}$ does not exist when $c = 0$. To do this, we will use the sequential

(cont'd) →

criterion for limits. Let $g(x) = \frac{f(x)-f(0)}{x-0}$ for short.

Then $g(x) = \frac{f(x)-0}{x-0}$ (because $f(0)=0$)

so $g(x) = \frac{f(x)}{x}$.

So $g(x) = \begin{cases} \frac{x}{x} & \text{if } x \in \mathbb{Q} \\ \frac{0}{x} & \text{if } x \notin \mathbb{Q} \end{cases}$

So $g(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$

(In fact, ~~the~~ $g(x)$ is the Dirichlet function we studied in class!)

Let (x_n) be any sequence of rational numbers s.t. $\lim(x_n) = 0$ and s.t. for all $n \in \mathbb{N}$, $x_n \neq 0$ ($(x_n) = (\frac{1}{n})$ will do.)
Then $g(x_n) = 1$ for all $n \in \mathbb{N}$.
Then $\lim(g(x_n)) = \lim(1) = 1$.

Let (y_n) be any sequence of irrational numbers s.t. $\lim(y_n) = 0$ and s.t. for all $n \in \mathbb{N}$, $y_n \neq 0$ ($(y_n) = (\frac{\sqrt{2}}{n})$ will do.)
Then $g(y_n) = 0$ for all $n \in \mathbb{N}$.
So $\lim(g(y_n)) = \lim(0) = 0$.

It follows from the sequential criterion that $\lim_{x \to 0} g(x)$ does not exist. QED

(c) Let (x_n) be any sequence of <u>rational</u> numbers such that $\lim(x_n) = c$ and ~~for~~ all $n \in \mathbb{N}$, $x_n \neq c$.
Then $f(x_n) = x_n$ for all $n \in \mathbb{N}$,
so $\lim(f(x_n)) = \lim(x_n) = c$.

(cont'd) →

Now let (y_n) be any sequence of irrational numbers such that $\lim(y_n) = c$ and for all $n \in \mathbb{N}$, $y_n \neq c$.

Then $f(y_n) = 0$ for all $n \in \mathbb{N}$,

so $\lim(f(y_n)) = \lim(0) = 0$.

If $c \neq 0$, then

$$\lim(f(x_n)) = c \neq 0 = \lim(f(y_n)).$$

Since $\lim(x_n) = \lim(y_n) = c$ and $\lim(f(x_n)) \neq \lim(f(y_n))$, the sequential criterion tells us that the limit of $f(x)$ does not exist as x approaches c.

Hence f is not continuous at c.