# Math 4163
# Assignment 9

**1.** Consider the heat kernel $K(x, y, t)$, defined for $t > 0$ and $x \in \mathbf{R}$, $y \in \mathbf{R}$ by

$$K(x, y, t) = \frac{1}{\sqrt{4\pi k t}} e^{-(x-y)^2/(4kt)}.$$

(a) Show that $K$ satisfies the heat equation,

$$K_t = k K_{xx}.$$

(b) Use L'hopital's rule to show that if $x \neq y$, then $\lim_{t \to 0^+} K(x, y, t) = 0$.

(c) Show that for all $t > 0$,

$$\int_{-\infty}^{\infty} K(x, y, t)\ dy = 1.$$

(You can use the fact that $\displaystyle\int_{-\infty}^{\infty} e^{-x^2}\ dx = \sqrt{\pi}$.)

**2.** Show that if $F(\omega)$ is the Fourier transform of $f(x)$, and $\beta$ is any real number, then the inverse Fourier transform of $e^{i\omega\beta} F(\omega)$ is $f(x - \beta)$.

**3.** Consider the initial-value problem for the heat equation with convection,

$$\begin{aligned} u_t &= k u_{xx} + c u_x && \text{for } -\infty < x < \infty \text{ and } t > 0, \\ u(x, 0) &= f(x) && \text{for } -\infty < x < \infty. \end{aligned}$$

where $k$ and $c$ are constants, and $f(x)$ is a given function.

(a) Apply the Fourier transform in $x$ to both sides of the equation, and use the properties of Fourier transforms of derivatives (see the inside back cover of the text) to obtain an ordinary differential equation in $t$ for $U(\omega, t)$, the Fourier transform of $u(x, t)$.

(b) Solve the ordinary differential equation for $U(\omega, t)$. The general solution will contain a function of $\omega$, which can be determined using the initial condition $u(x, 0) = f(x)$.

(c) Take the inverse Fourier transform of $U(\omega, t)$ to obtain an expression for $u(x, t)$. Use the result of problem **2** above. Your answer should give $u(x, t)$ as an integral involving $f$.

**4.** Consider the initial-value problem for Laplace's equation on an infinite strip:

$$\begin{aligned} u_{xx} + u_{yy} &= 0 && \text{for } -\infty < x < \infty \text{ and } 0 < y < L, \\ u(x, 0) &= 0 && \text{for } -\infty < x < \infty, \\ u(x, L) &= f(x) && \text{for } -\infty < x < \infty, \end{aligned}$$

where $f(x)$ is a given function.

(a) Apply the Fourier transform in $x$ to both sides of the equation, and use the properties of Fourier transforms of derivatives (see the inside back cover of the text) to obtain an ordinary differential equation in $y$ for $U(\omega, y)$, the Fourier transform of $u(x, y)$.

(b) Solve the ordinary differential equation for $U(\omega, y)$. The general solution will contain two functions of $\omega$, which can be determined using the boundary conditions $u(x, 0) = 0$ and $u(x, L) = f(x)$.

(c) Take the inverse Fourier transform of $U(\omega, y)$ to obtain an expression for $u(x, y)$. Use the fact that if $G(\omega) = \dfrac{\sinh(\omega a)}{\sinh(\omega b)}$, where $a$ and $b$ are real numbers with $0 < a < b$, then the inverse Fourier transform of $G(\omega)$ is

$$g(x) = \frac{\pi}{b} \left( \frac{\sin\left(\frac{\pi a}{b}\right)}{\cosh\left(\frac{\pi x}{b}\right) + \cos\left(\frac{\pi a}{b}\right)} \right).$$

Your answer should give $u(x, y)$ as an integral involving $f$.