Math 4163
Solutions to problems on Assignment 6

1. As we saw when we solved Laplace's equation on the entire circle, separated solutions $v(r,\theta) = \phi(\theta)G(r)$ of $v_{rr} + \frac{1}{r}v_r + \frac{1}{r^2}v_{\theta\theta}$ satisfy the two ODE's $\phi''(\theta) + \lambda\phi(\theta) = 0$ and $r^2G''(r) + rG'(r) - \lambda G(r) = 0$, where λ is a constant. If the separated solution is to satisfy the the boundary conditions $v(r,0) = v(r,\theta/2) = 0$ for $0 \le r \le 1$, we see that for the solution to be nontrivial, we must have $\phi(0) = \phi(\theta/2) = 0$.

We already know from previous work that the eigenvalue problem $\phi'' + \lambda\phi = 0$, $\phi(0) = \phi(L) = 0$, has eigenvalues $\lambda = (n\pi/L)^2$, $n \in \{1, 2, 3, \dots\}$, with corresponding eigenfunctions $\phi(\theta) = \sin(n\pi\theta/L)$. In this case we have $L = \pi/2$, so the eigenvalues are $\lambda = (2n)^2$, $n \in \{1, 2, 3, \dots\}$, and the corresponding eigenfunctions are $\phi(\theta) = \sin(2n\theta)$.

We also know from previous work that for $\lambda > 0$, the solution of the ODE for $G(r)$ can be found by putting $G(r) = r^p$ and finding two independent solutions by solving a quadratic equation for p. This gives us the general solution of the ODE for G as

$$G(r) = Ar^{\sqrt{\lambda}} + Br^{-\sqrt{\lambda}},$$

and in this case, since $\lambda = (2n)^2$,

$$G(r) = Ar^{2n} + Br^{-2n}.$$

If the separated solution $v(r,\theta) = \phi(\theta)G(r)$ is to satisfy the boundary condition that $\lim_{r\to 0} v(r,\theta)$ exists, we see that we must have $B = 0$, so $G(r) = Ar^{2n}$, and

$$v(r,\theta) = r^{2n}\sin(2n\theta).$$

Now we look for a linear combination

$$v(r,\theta) = \sum_{n=1}^{\infty} B_n r^{2n}\sin(2n\theta)$$

which satisfies $v_r(1,\theta) = 1$ for $0 \le \theta \le \pi/2$. We have

$$v_r(r,\theta) = \sum_{n=1}^{\infty} B_n 2nr^{2n-1}\sin(2n\theta),$$

so we want

$$v_r(1,\theta) = \sum_{n=1}^{\infty} 2nB_n\sin(2n\theta) = 1.$$

In other words, the numbers $2nB_n$ should be the coefficients in the Fourier sine series for the function $f(\theta) = 1$ on $[0, \pi/2]$. From the formula for the coefficients in a Fourier sine series on $[0, L]$, we have, with $L = \pi/2$,

$$2nB_n = \frac{2}{L}\int_0^L f(\theta)\sin(n\pi\theta/L)\ d\theta = \frac{4}{\pi}\int_0^{\pi/2}\sin(2n\theta)\ d\theta = \frac{4}{2n\pi}\left(-\cos(\pi n) + 1\right),$$

so

$$B_n = \frac{1}{\pi n^2}\left(1 - \cos(\pi n)\right)$$

and

$$v(r,\theta) = \frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1-\cos(\pi n)}{n^2}\ r^{2n}\sin(2n\theta).$$

2. As in the solution of Laplace's equation on the circle, we have that separated solutions $v(r,\theta) = \phi(\theta)G(r)$ of $v_{rr} + \frac{1}{r}v_r + \frac{1}{r^2}v_{\theta\theta}$ satisfy the two ODE's $\phi''(\theta) + \lambda\phi(\theta) = 0$ and $r^2G''(r) + rG'(r) - \lambda G(r) = 0$; and $\phi(\theta)$ satisfies periodic boundary conditions $\phi(-\pi) = \phi(\pi)$ and $\phi'(-\pi) = \phi'(\pi)$. We already know that the eigenvalues of the problem for $\phi(\theta)$ are $\lambda = 0$, with corresponding eigenfunction $\phi(\theta) = 1$, and $\lambda = (n\pi)^2$, $n \in \{1, 2, 3, \dots\}$, with corresponding eigenfunctions $\phi(\theta) = \cos(n\theta)$ and $\phi(\theta) = \sin(n\theta)$. We also know that independent solutions of the equation for $G(r)$ are

$$G(r) = 1 \quad \text{and} \quad G(r) = \log r,$$

when $\lambda = 0$, and

$$G(r) = r^n \quad \text{and} \quad G(r) = r^{-n},$$

when $\lambda = (n\pi)^2$.

So we take

$$v(r,\theta) = A_0 + C_0 \log r + \sum_{n=1}^{\infty} \big(A_n r^n \cos(n\theta) + B_n r^n \sin(n\theta) + C_n r^{-n} \cos(n\theta) + D_n r^{-n} \sin(n\theta)\big).$$

To satisfy $v(3,\theta) = 0$, we need

$$0 = A_0 + C_0 \log 3 + \sum_{n=1}^{\infty} (3^n A_n + 3^{-n} C_n) \cos(n\theta) + (3^n B_n + 3^{-n} D_n) \sin(n\theta).$$

The coefficients of the Fourier series for the constant function 0 are all zero, so we must have $A_0 + C_0 \log 3 = 0$ and $3^n A_n + 3^{-n} C_n = 0$ and $3^n B_n + 3^{-n} D_n = 0$ for all $n \in \{1, 2, 3, \dots\}$. Hence $A_0 = -C_0 \log 3$ and $A_n = -C_n/9^n$, and $B_n = -D_n/9^n$ for all $n \in \{1, 2, 3, \dots\}$. This then gives

$$v(r,\theta) = C_0 \log(r/3) + \sum_{n=1}^{\infty} C_n (r^{-n} - (r/9)^n) \cos(n\theta) + D_n (r^{-n} - (r/9)^n) \sin(n\theta).$$

To satisfy $v(4,\theta) = f(\theta)$, we need

$$f(\theta) = C_0 \log(4/3) + \sum_{n=1}^{\infty} C_n (4^{-n} - (4/9)^n) \cos(n\theta) + D_n (4^{-n} - (4/9)^n) \sin(n\theta),$$

so the coefficients on the right of the equation must be the coefficients in the Fourier series for $f(\theta)$ on $[-\pi, \pi]$. Thus

$$\begin{aligned}
C_0 \log(4/3) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\theta)\, d\theta = \frac{1}{2\pi} \int_0^{\pi} 1\, d\theta = \frac{1}{2}, \\
C_n (4^{-n} - (4/9)^n) &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \cos(n\theta)\, d\theta = \frac{1}{\pi} \int_0^{\pi} \cos(n\theta)\, d\theta = 0, \\
D_n (4^{-n} - (4/9)^n) &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(\theta) \sin(n\theta)\, d\theta = \frac{1}{\pi} \int_0^{\pi} \sin(n\theta)\, d\theta = \frac{1 - \cos(n\pi)}{n\pi},
\end{aligned}$$

which gives us that $C_0 = 1/(2\log(4/3))$, and $C_n = 0$ and $D_n = \dfrac{1 - \cos(n\pi)}{n\pi(4^{-n} - (4/9)^n)}$ for $n \geq 1$. Therefore

$$v(r,\theta) = \frac{\log(r/3)}{2\log(4/3)} + \sum_{n=1}^{\infty} \left(\frac{1 - \cos(n\pi)}{n\pi}\right) \left(\frac{r^{-n} - (r/9)^n}{4^{-n} - (4/9)^n}\right) \sin(n\theta).$$

3.

(a) For $\lambda = 0$, the general solution of the ODE $\phi''(x) = 0$ is $\phi(x) = A + Bx$, with derivative $\phi'(x) = B$. The condition $\phi(0) = \phi'(0)$ then gives $A = B$, so $\phi(x) = A + Ax$ and $\phi'(x) = A$. But then the condition $\phi(1) = -\phi'(1)$ gives $A + A = -A$, which implies $A = 0$ and hence $\phi(x) \equiv 0$. So, for $\lambda = 0$, the only function ϕ which solves the problem is the trivial solution $\phi(x) \equiv 0$. Hence $\lambda = 0$ is not an eigenvalue.

(b) The Rayleigh quotient is the following relation between an eigenvalue λ and its corresponding eigenfunction $\phi(x)$:

$$\lambda = \frac{\int_0^1 \phi'(x)^2\ dx - \phi(1)\phi'(1) + \phi(0)\phi'(0)}{\int_0^1 \phi(x)^2\ dx}.$$

For this problem, using the conditions $\phi(0) = \phi'(0)$ and $\phi(1) = -\phi'(1)$, we can rewrite the Rayleigh quotient as

$$\lambda = \frac{\int_0^1 \phi'(x)^2\ dx + \phi'(1)^2 + \phi'(0)^2}{\int_0^1 \phi(x)^2\ dx}.$$

Since the square of a real number is always non-negative, then the numerator of this expression is the sum of non-negative terms, so is non-negative; and the denominator is also non-negative. So their quotient λ is also non-negative.

(c) For $\lambda \geq 0$, the general solution of the ODE for ϕ is

$$\phi(x) = A\cos\sqrt{\lambda}x + B\sin\sqrt{\lambda}x,$$

and its derivative is

$$\phi'(x) = -A\sqrt{\lambda}\sin\sqrt{\lambda}x + B\sqrt{\lambda}\cos\sqrt{\lambda}x.$$

The condition $\phi(0) = \phi'(0)$ gives $A = B\sqrt{\lambda}$, so

$$\phi(x) = B(\sqrt{\lambda}\cos\sqrt{\lambda}x + \sin\sqrt{\lambda}x)$$

and

$$\phi'(x) = B(-\lambda\sin\sqrt{\lambda}x + \sqrt{\lambda}\cos\sqrt{\lambda}x).$$

Then the condition $\phi(1) = -\phi'(1)$ gives

$$\sqrt{\lambda}\cos\sqrt{\lambda} + \sin\sqrt{\lambda} = \lambda\sin\sqrt{\lambda} - \sqrt{\lambda}\cos\sqrt{\lambda},$$

or

$$2\sqrt{\lambda}\cos\sqrt{\lambda} = (\lambda - 1)\sin\sqrt{\lambda}.$$

Dividing by $\cos\sqrt{\lambda}$ and $\lambda - 1$ gives the desired equation.

(The graphs of $\tan\sqrt{\lambda}$ and $\frac{2\sqrt{\lambda}}{\lambda-1}$, with a few points of intersection marked, should be included here.)

4. For $\lambda = -\kappa^2$, the general solution of the ODE for ϕ is $\phi(x) = A\cosh\kappa x + B\sinh\kappa x$, and the condition $\phi(0) = 0$ gives $A = 0$, so $\phi(x) = B\sinh\kappa x$ and $\phi'(x) = B\kappa\cosh\kappa x$. Then the condition $\phi'(1) = 10\phi(1)$ becomes

$$B\kappa\cosh\kappa = 10B\sinh\kappa.$$

This yields the equation

$$\tanh\kappa = \frac{\kappa}{10},$$

which can be seen to have a positive solution by graphing $\tanh\kappa$ and $\kappa/10$ as functions of κ. A corresponding eigenfunction is $\phi(x) = \sinh\kappa x$.

(The graphs of $\tanh\kappa$ and $\kappa/10$ should be included here as part of your answer.) The reason that the graphs intersect at some point where $\kappa > 0$ is that the graph of $\kappa/10$ starts out below the graph of $\tanh\kappa$ for κ near zero, because the derivative of $\tanh\kappa$ is 1 at $\kappa = 0$, so the slope of $\tanh\kappa$ is greater than that of $\kappa/10$ at $\kappa = 0$. But for, say, $\kappa = 100$, $\tanh\kappa$ is near 1 and $\kappa/10 = 10$, so there the graph of $\kappa/10$ is above that of $\tanh\kappa$. So somewhere in between $\kappa = 0$ and $\kappa = 10$, the graphs must cross.

5. Putting $u(x,t) = \phi(x)G(t)$ into the PDE and separating variables, we get

$$\frac{G''(t) + \beta G'(t)}{c^2 G(t)} = \frac{\phi''(x)}{\phi(x)} = -\lambda,$$

which gives us the ODEs $\phi''(x) + \lambda\phi(x) = 0$ and

$$G''(t) + \beta G'(t) + c^2\lambda G(t) = 0.$$

The boundary conditions on u at $x = 0$ and $x = 1$ give us that, for nontrivial solutions, $\phi(0) = \phi(1) = 0$. As we already know, the eigenvalues for this eigenvalue problem for ϕ are $\lambda = (n\pi)^2$, with $n \in \{1, 2, 3, \dots\}$, and the corresponding eigenfunctions are $\phi(x) = \sin(n\pi x)$.

Putting $\lambda = (n\pi)^2$ into the ODE for G gives

$$G''(t) + \beta G'(t) + (cn\pi)^2 G(t) = 0.$$

This is a linear ODE with constant coefficients, so solutions can be found by putting $G(t) = e^{rt}$ into the equation and solving for r. We get

$$r^2 + \beta r + (cn\pi)^2 = 0,$$

whose solutions are

$$r = \frac{-\beta + \sqrt{\beta^2 - 4c^2n^2\pi^2}}{2} \quad \text{and} \quad r = \frac{-\beta - \sqrt{\beta^2 - 4c^2n^2\pi^2}}{2}.$$

We were given that $\beta < 4n^2\pi^2c^2$, but as you might have guessed, this was a typo: the condition should have been $\beta^2 < 4n^2\pi^2c^2$. Under this condition, the quantity under the radical in the formula will be negative, so there will be two different solutions to the ODE, given by

$$G(t) = e^{-\beta t/2} e^{\pm it\sqrt{4n^2\pi^2c^2 - \beta^2}},$$

or more conveniently,

$$G(t) = e^{-\beta t/2}\cos(\gamma_n t) \quad \text{and} \quad e^{-\beta t/2}\sin(\gamma_n t),$$

where $\gamma_n = \frac{1}{2}\sqrt{4n^2\pi^2c^2 - \beta^2}$. So we can take

$$u(x,t) = \sum_{n=1}^{\infty} \sin(n\pi x) e^{-\beta t/2}\left[A_n\cos(\gamma_n t) + B_n\sin(\gamma_n t)\right], \tag{1}$$

and hence

$$u_t(x,t) = \sum_{n=1}^{\infty} \sin(n\pi x)\left(\frac{-\beta}{2} e^{-\beta t/2}\left[A_n\cos(\gamma_n t) + B_n\sin(\gamma_n t)\right] + e^{-\beta t/2}\gamma_n\left[-A_n\sin(\gamma_n t) + B_n\cos(\gamma_n t)\right]\right).$$

Substituting $t = 0$ into these equations and using the given initial conditions, we get, for all $x \in [0,1]$,

$$f(x) = \sum_{n=1}^{\infty} A_n \sin(n\pi x)$$

and

$$g(x) = \sum_{n=1}^{\infty}\left(\frac{-\beta}{2}A_n + \gamma_n B_n\right)\sin(n\pi x).$$

From the first of these equations, we get

$$A_n = 2\int_0^1 f(x)\sin(n\pi x)\, dx, \tag{2}$$

and from the second we get

$$\frac{-\beta}{2} A_n + \gamma_n B_n = 2 \int_0^1 g(x) \sin(n\pi x)\ dx.$$

Therefore

$$B_n = \frac{\beta A_n}{2\gamma_n} + \frac{2}{\gamma_n} \int_0^1 g(x) \sin(n\pi x)\ dx.$$

Using the formulas for A_n and γ_n from above, this gives

$$B_n = \frac{4}{\sqrt{4n^2\pi^2c^2 - \beta^2}} \int_0^1 \left(\frac{\beta}{2} f(x) + g(x) \right) \sin(n\pi x)\ dx. \tag{3}$$

The solution to the problem is the series given in (1) for $u(x,t)$, with coefficients A_n and B_n given in (2) and (3).