Math 4163
Assignment 6

For problems 1 and 2 below, write $v(r,\theta) = G(r)\phi(\theta)$ and use separation of variables. You can use the solutions to the eigenvalue problem for ϕ and the ordinary differential equation for G which we have found in class without having to rederive them here. Your answer will be in the form of a series for $v(r,\theta)$.

1. Find the solution $v(r,\theta)$ of the following boundary-value problem for Laplace's equation on the quarter-circle:

$$\begin{aligned} v_{rr} + \frac{1}{r}v_r + \frac{1}{r^2}v_{\theta\theta} &= 0 \qquad \text{for } 0 \le r < 1 \text{ and } 0 < \theta < \pi/2, \\ v(r,0) &= 0 \qquad \text{for } 0 \le r \le 1, \\ v(r,\frac{\pi}{2}) &= 0 \qquad \text{for } 0 \le r \le 1, \\ v_r(1,\theta) &= 1 \qquad \text{for } 0 \le \theta \le \frac{\pi}{2}, \\ \lim_{r\to 0} v(r,\theta) &\text{ exists} \qquad \text{for } 0 \le \theta \le \frac{\pi}{2}. \end{aligned}$$

2. Find the solution $v(r,\theta)$ of the following boundary-value problem for Laplace's equation on the annulus:

$$\begin{aligned} v_{rr} + \frac{1}{r}v_r + \frac{1}{r^2}v_{\theta\theta} &= 0 \qquad \text{for } 3 < r < 4 \text{ and } -\pi \le \theta < \pi, \\ v(3,\theta) &= 0 \qquad \text{for } -\pi \le \theta < \pi, \\ v(4,\theta) &= f(\theta) \qquad \text{for } -\pi \le \theta < \pi, \end{aligned}$$

where

$$f(\theta) = \begin{cases} 0 & \text{for } -\pi \le \theta \le 0 \\ 1 & \text{for } 0 < \theta < \pi. \end{cases}$$

3. Consider the boundary-value problem

$$\begin{aligned} \phi''(x) + \lambda\phi(x) &= 0 \qquad \text{for } 0 \le x \le 1, \\ \phi(0) &= \phi'(0) \\ \phi(1) &= -\phi'(1). \end{aligned}$$

(a) Show that $\lambda = 0$ is not an eigenvalue.
(b) Use the Rayleigh quotient to show that $\lambda \ge 0$.
(c) Show that the positive eigenvalues are the solutions of the equation

$$\tan\sqrt{\lambda} = \frac{2\sqrt{\lambda}}{\lambda - 1},$$

and, by graphing both sides of this equation and seeing where they intersect, draw a diagram showing the first few eigenvalues.

4. Show that for the problem

$$\begin{aligned} \phi''(x) + \lambda\phi(x) &= 0 \qquad \text{for } 0 \le x \le 1, \\ \phi(0) &= 0 \\ \phi'(1) &= 10 \cdot \phi(1) \end{aligned}$$

there is a negative eigenvalue $\lambda = -\kappa^2$, by finding an equation for κ and showing graphically that the equation has a solution. Also give a formula for the corresponding eigenfunction $\phi(x)$.

5. Find the solution of the problem

$$\begin{aligned} u_{tt} + \beta u_t &= c^2 u_{xx} \qquad \text{for } 0 < x < 1 \text{ and } t > 0, \\ u(0,t) &= 0 \qquad \text{for } t \geq 0, \\ u(1,t) &= 0 \qquad \text{for } t \geq 0, \\ u(x,0) &= f(x) \qquad \text{for } 0 \leq x \leq 1, \\ u_t(x,0) &= g(x) \qquad \text{for } 0 \leq x \leq 1. \end{aligned}$$

You can assume that $0 < \beta < 4\pi^2 c^2$ (so also $\beta < 4n^2\pi^2 c^2$ for all $n = 1, 2, 3, \ldots$). Your answer will be a series for $u(x,t)$ with coefficients which are given in terms of integrals involving $f(x)$ and $g(x)$.