EXAM 2
Math 4103
4-1-11

Name key

1. *(10 points)* Let $f(z) = \frac{z^2 - z}{z+8}$. Show that $\lim_{z\to\infty} f(z) = \infty$ by evaluating $\lim_{z\to 0} \frac{1}{f(1/z)}$.

$$\lim_{z\to 0} \frac{1}{f(1/z)} = \lim_{z\to 0} \frac{1}{\left(\frac{(\frac{1}{z})^2 - (\frac{1}{z})}{(\frac{1}{z})+8}\right)} \;②\; = \lim_{z\to 0} \frac{\frac{1}{z}+8}{(\frac{1}{z})^2 - (\frac{1}{z})} \;②\; = \lim_{z\to 0} \frac{z^2(\frac{1}{z}+8)}{z^2(\frac{1}{z^2}-\frac{1}{z})} \;②\; = \lim_{z\to 0} \frac{z+8z^2}{1-z} \;②$$

$$= \frac{0+0}{1-0} = 0. \;②$$

(Since $\lim_{z\to 0} \frac{1}{f(1/z)} = 0$, then $\lim_{z\to\infty} f(z) = \infty$.)

2. *(12 points)* Let $f(z) = \text{Re}(z)$. Show that f is nondifferentiable at every z by showing that the limit $\lim_{\Delta z\to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$ does not exist.

Let $\Delta z = \Delta x + i\Delta y$, and consider the case when $\Delta y = 0$. Then

$$\lim_{\substack{\Delta z\to 0\\ \Delta y = 0}} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\substack{\Delta z\to 0\\ \Delta y=0}} \frac{\text{Re}(z+\Delta z) - \text{Re}\, z}{\Delta z} \;②\; = \lim_{\Delta x\to 0} \frac{(x+\Delta x) - x}{\Delta x} \;②\; = \lim_{\Delta x\to 0} \frac{\Delta x}{\Delta x} = 1. \;②$$

Now consider the case when $\Delta x = 0$. Then

$$\lim_{\substack{\Delta z\to 0\\ \Delta x = 0}} \frac{f(z+\Delta z) - f(z)}{\Delta z} = \lim_{\substack{\Delta z\to 0\\ \Delta x=0}} \frac{\text{Re}(z+\Delta z) - \text{Re}\, z}{\Delta z} = \lim_{\Delta y\to 0} \frac{x+0-x}{i\Delta y} \;②\; = \lim_{\Delta y\to 0} \frac{0}{i\Delta y} = 0. \;②$$

Since the two limits above are not equal, ② $\lim_{\Delta z\to 0} \frac{f(z+\Delta z) - f(z)}{\Delta z}$ does not exist.

3. *(15 points)* Define $f(x+iy) = (3y^3 - 9x^2y - 2xy) + i(x^2 - y^2 + 3x^3 - 9xy^2)$.

a. Show that f is entire by checking that the Cauchy-Riemann equations hold on $\mathbf{C}$.

[11] Here $u = 3y^3 - 9x^2y - 2xy$, so $u_x = -18xy - 2y$ ②
and $u_y = 9y^2 - 9x^2 - 2x$ ②

Also $v = x^2 - y^2 + 3x^3 - 9xy^2$, so $v_x = 2x + 9x^2 - 9y^2$ ②
and $v_y = -2y - 18xy$ ②

So $u_x = v_y$ and $u_y = -v_x$. ③ Therefore the CR equations hold on $\mathbb{C}$, so f is entire.

b. Find a formula for $f'(x+iy)$.

[4] $$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} \;②\; = (-18xy - 2y) + i(2x + 9x^2 - 9y^2) \;②$$

[By the way, from this formula we see that $f'(z) = 2iz + 9iz^2$,
Also $f(z) = iz^2 + 3iz^3$.]

4. *(25 points)* **Let $f(x+iy) = x^4 + iy^4$.**

a. Show that f is differentiable at every point where $x = y$.

[10] Here $u = x^4$ and $v = y^4$. (2)

Thus u and v are C^1 functions on $\mathbb{C}$, (2) and $u_x = 4x^3$, $v_x = 0$ (2)
$u_y = 0$, $v_y = 4y^3$

At any point where $x = y$, we have $u_x = v_y$ (since $4x^3 = 4y^3$) (2)
and $u_y = -v_x$ (since $0 = -0$),

so the CR eqns. hold. Since u and v are C^1, it follows from a theorem in class that f is differentiable at that point. (2)

b. Show that f is not differentiable at any point where $x \neq y$.

[10] If $x \neq y$, then since $u_x = 4x^3$ and $v_y = 4y^3$, and $x \neq y \Rightarrow x^3 \neq y^3 \Rightarrow 4x^3 \neq 4y^3$, it follows that $u_x \neq v_y$, so the CR equations do not hold at that point (5)

Since the CR equations are a necessary condition for differentiability, then f is not differentiable at that point (5)

c. Is f analytic at $z = 0$? Explain.

[5] No. In any neighborhood of 0, there exists points where $x \neq y$, and f is not differentiable at those points. Therefore the statement "there exists a neighborhood of 0 s.t. f is differentiable at all points in the neighborhood" is false.

5. *(10 points)* **Define $f(re^{i\theta}) = \left(r^2 - \frac{1}{r^2}\right)\cos(2\theta) + i\left(r^2 + \frac{1}{r^2}\right)\sin(2\theta)$. Show that f is analytic on $\mathbf{C} - \{0\}$ by checking that the Cauchy-Riemann equations in polar coordinates hold when $r > 0$.**

Here $u = \left(r^2 - \frac{1}{r^2}\right)\cos 2\theta$, so $u_r = \left(2r + \frac{2}{r^3}\right)\cos 2\theta$ (2)

and $u_\theta = \left(r^2 - \frac{1}{r^2}\right)(-2 \sin 2\theta)$ (2)

and $v = \left(r^2 + \frac{1}{r^2}\right)\sin 2\theta$, so $v_r = \left(2r - \frac{2}{r^3}\right)\sin 2\theta$ (1)

and $v_\theta = \left(r^2 + \frac{1}{r^2}\right) 2\cos 2\theta$. (2)

So $ru_r = 2\left(r^2 + \frac{1}{r^2}\right)\cos 2\theta = v_\theta$ (2)

and $rv_r = 2\left(r^2 - \frac{1}{r^2}\right)\sin 2\theta = -u_\theta$.

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6. *(20 points)* Let $u(x,y) = (e^{2y} + e^{-2y}) \sin 2x$.

a. Show that u is harmonic on C.

[10] $u_x = (e^{2y} + e^{-2y}) \cdot 2\cos 2x$ 2 $\quad u_y = (2e^{2y} - 2e^{-2y}) \sin 2x$ 2

$u_{xx} = (e^{2y} + e^{-2y}) \cdot (-4\sin 2x)$ 2 $\quad u_{yy} = (4e^{2y} + 4e^{-2y}) \sin 2x$ 2

So $u_{xx} = -u_{yy}$, so $u_{xx} + u_{yy} = 0$ on $\mathbb{C}$.

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b. Find a harmonic conjugate v of u on C.

[10] We must have $v_y = u_x =$ 2 $2e^{2y}\cos 2x + 2e^{-2y}\cos 2x$

So $v = \int^y u_x = e^{2y}\cos 2x - e^{-2y}\cos 2x + \varphi(x)$ 2

~~Also~~ Hence $v_x = e^{2y}(-2\sin 2x) + e^{-2y}\, 2\sin 2x + \varphi'(x)$ 4

For $v_x = -u_y$ we must therefore have

$v_x = -2e^{2y}\sin 2x + 2e^{-2y}\sin 2x + \varphi'(x) = -(2e^{2y} - 2e^{-2y})\sin 2x = -u_y$

and comparing the two functions in the middle of the equation, we see $\varphi'(x) = 0$,

so $\varphi(x) = C$ (constant). Thus $v = e^{2y}\cos 2x - e^{-2y}\cos 2x + C$, where C is any constant
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7. *(8 points)* **Suppose u is harmonic on C, and v is a harmonic conjugate of u. Show that $e^u \cos v$ is harmonic on C, and $e^u \sin v$ is a harmonic conjugate of $e^u \cos v$. (Hint: what can you say about $f = u + iv$?)**

Since u is harmonic and v is a harmonic conjugate of u on $\mathbb{C}$, Then $f = u + iv$ is analytic on $\mathbb{C}$ (entire) 2

By the chain rule, since f is analytic on $\mathbb{C}$,

so is $g(z) = e^{(f(z))}$ 2

But then $g(z) = e^{(u+iv)} = (e^u \cos v) + i(e^u \sin v)$ 2

Since g is analytic on $\mathbb{C}$, it follows that $e^u \cos v$ and $e^u \sin v$ are harmonic on $\mathbb{C}$, and $e^u \sin v$ is a harmonic conjugate of $e^u \cos v$. 2