Math 4073 — Numerical Analysis
Assignment 3

1. A cubic spline S for a function f is given by

$$S(x) = \begin{cases} 1 + x + 2x^2 & \text{for } 0 \le x \le 1, \\ a + b(x-1) + c(x-1)^2 + d(x-1)^3 & \text{for } 1 \le x \le 2. \end{cases}$$

We require the cubic spline to be a *clamped* cubic spline, which means that the derivatives of S should equal the derivatives of f at the points $x = 0$ and $x = 2$. We are given that $f'(0) = 1$ and $f'(2) = 0$.

Find the values of the constants a, b, c, and d.

2. Find values of the constants A, B, and C so that the numerical differentiation formula for $f'(x_0)$,

$$f'(x_0) = \left(\frac{Af(x_0) + Bf(x_0 + h) + Cf(x_0 + 2h)}{h} \right) + O(h^2),$$

is valid for all sufficiently differentiable functions f.

(Hint: expand $f(x_0 + h)$ and $f(x_0 + 2h)$ in Taylor series about x_0 up to order h^3, and substitute into the left-hand side of the equation

$$Af(x_0) + Bf(x_0 + h) + Cf(x_0 + 2h) - hf'(x_0) - O(h^3) = 0.$$

In the resulting expression, collect the terms according to powers of h and set each of the coefficients of h^0, h^1, and h^2 equal to zero. This will give you three equations which you can solve for A, B, and C.)

3. The approximation $(1 + h)^{1/h} \approx e$ has error term given by

$$(1 + h)^{1/h} = e + K_1 h + K_2 h^2 + K_3 h^3 + \dots$$

as $h \to 0$, where K_1, K_2, $\dots$ are independent of h. Use Richardson extrapolation to find an approximation to e with error $O(h^2)$; and apply Richardson extrapolation again to find an approximation to e with error $O(h^3)$.

4. Determine the values of N and $h = (2 - 1)/N$ required to approximate $\displaystyle\int_1^2 x \ln x \, dx$ to within 10^{-5}, and compute the approximation. Use:

(a) the composite Trapezoid Rule.
(b) the composite Simpson's Rule.

5.

(a) Use the method of undetermined coefficients to find constants A, B, C, and D so that the Newton-Cotes approximation formula

$$\int_0^1 f(x)\, dx \approx Af(0) + Bf(1/3) + Cf(2/3) + Df(1)$$

is exact for all polynomials of degree three or less.

(b) Use your answer from part (a) to write down the Newton-Cotes approximation formula for the integral of $f(x)$ on the interval $[x_0, x_0 + 3h]$, using the points $\{x_0, x_0 + h, x_0 + 2h, x_0 + 3h\}$. What power of h do you expect to appear in the error term, and why?