Math 4073 — Numerical Analysis
Assignment 1

1. Suppose you want to find the positive solution of the equation $x = 3\sin x$ (note: we don't want the negative solution or the solution $x = 0$). Choose an appropriate starting interval $[a, b]$, and use the error estimate for the bisection method to estimate how many iterations this method would take to approximate the solution to within an error of 10^{-6} using your choice of a and b. Use the MATLAB program bisection.m to carry out this many steps and obtain the approximate solution. Include a printout of the successive approximations computed.

2. Suppose you want to find the solution of the equation $x^4 - 2x^2 - 3$ in the interval $[1, 2]$.

a) Rewrite the equation in the form $x = g(x)$ where $g(x)$ is a function satisfying the inequality $|g'(x)| \le k$ for all $x \in [1, 2]$ and k is some number less than 1. Justify your answer by giving the value of k and explaining how you know that the inequality is satisfied.

b) Use the error estimate for the fixed-point iteration method to estimate how many iterations this method would take to approximate the solution to within an error of 10^{-6} using $p_0 = 1$. Use the MATLAB progam fixed_point.m to carry out this many steps and obtain the approximate solution. Include a printout of the successive approximations computed.

3. This problem considers a method for approximating $\sqrt{2}$.

a) Use the theorem on convergence of the fixed-point iteration method to show that the sequence defined by

$$p_n = \frac{1}{2}p_{n-1} + \frac{1}{p_{n-1}}$$

converges to $\sqrt{2}$ as long as p_0 is chosen to be greater than $\sqrt{2}$. (You can do this by taking the interval $[a, b]$ in the theorem to be $[\sqrt{2}, p_0]$.)

b) Show that if p_0 is any number such that $0 < p_0 < \sqrt{2}$, then the number p_1 given by the above formula will satisfy $p_1 > \sqrt{2}$. (Hint: perhaps the easiest way to do this is to show that the inequality $p_1 > \sqrt{2}$ can be rewritten in the form $(p_0 - \sqrt{2})^2 > 0$, which is obviously true because the square of any non-zero number is positive.) From this conclude that the fixed-point method converges to $\sqrt{2}$ if p_0 is any positive number whatsoever.

4. Show that Newton's method for finding a root p of the equation $f(x) = 0$ can be rewritten as a fixed-point iteration method for finding a fixed point p of the equation $x = g(x)$, where $g'(p) = 0$. Identify the function $g(x)$ and prove that if $f(p) = 0$ then $g'(p) = 0$.

5. Use Newton's method, with appropriate choices of the initial guess p_0, to find the two positive solutions of the equation $4x^2 - e^x - e^{-x} = 0$ to within 10^{-5}. Use newton.m or newton_raphson.m, and include a printout of the successive approximations computed.