Solutions to problems on Assignment 11

Problem 3, section 9.5.
I solved this in class on Wednesday, using a couple of different methods.

Problem 9, section 9.5.
We have to find the solution to $10u_t = u_{xx}$ on $0 < x < 5$, $t > 0$; satisfying the boundary conditions $u(0, t) = u(5, t) = 0$ for $t > 0$; and the initial condition $u(x, 0) = 25$ for $0 < x < 5$.

For this problem $k = 1/10$ (since the equation can be rewritten as $u_t = (0.1)u_{xx}$; $L = 5$; and the initial temperature profile is the constant function $f(x) = 25$. According to Theorem 1 on page 622, the solution is given by

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-n^2\pi^2(\frac{1}{10})t/5^2} \sin(n\pi x/5),$$

or

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-n^2\pi^2 t/250} \sin(n\pi x/5),$$

where B_n is given by

$$B_n = \frac{2}{5}\int_0^5 25\sin\left(\frac{n\pi x}{5}\right)\, dx.$$

To compute this integral, make the substitution $u = n\pi x/5$, so $du = \frac{n\pi}{5}\, dx$ and $dx = \frac{5}{n\pi}\, du$. The limits of the integral change from $x = 0$ and $x = 5$ to $u = n\pi \cdot 0/5 = 0$ and $u = n\pi \cdot 5/5 = n\pi$, and we get

$$\begin{aligned}
B_n &= \frac{2}{5}\int_0^{n\pi} 25\sin u\left(\frac{5}{n\pi}\, du\right) \\
&= \frac{50}{n\pi}\int_0^{n\pi} \sin u\, du = \frac{50}{n\pi}\left(-\cos u|_0^{n\pi}\right) = \frac{50}{n\pi}\left(-\cos n\pi + 1\right) \\
&= \begin{cases} \frac{100}{n\pi} & (n \text{ odd}) \\ 0 & (n \text{ even}). \end{cases}
\end{aligned}$$

Therefore

$$\begin{aligned}
u(x,t) &= \sum_{n \text{ odd}} \frac{100}{n\pi} e^{-n^2\pi^2 t/250} \sin(n\pi x/5) \\
&= \frac{100}{\pi}\left(\sin(\pi x/5)e^{-\pi^2 t/250} + \frac{1}{3}\sin(3\pi x/5)e^{-9\pi^2 t/250} + \frac{1}{5}\sin(5\pi x/5)e^{-25\pi^2 t/250} + \dots\right).
\end{aligned}$$

Problem 13, section 9.5.
This problem is similar to problem 9, the only differences being that here $L = 40$ cm, the initial temperature profile is given by the constant function $f(x) = 100$, and the value of k will be different (depending on what material the rod is made of).

(a) As in the solution of problem 9 above, we get

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-n^2\pi^2 kt/1600} \sin(n\pi x/40),$$

where B_n is given by

$$B_n = \frac{2}{40}\int_0^{40} 100\sin\left(\frac{n\pi x}{40}\right)\, dx.$$

Computing B_n as before, we find that

$$B_n = \begin{cases} \frac{400}{n\pi} & (n \text{ odd}) \\ 0 & (n \text{ even}). \end{cases}$$

(b) From part (a), we have

$$u(x,t) = \frac{400}{\pi}\left(\sin(\pi x/40)e^{-\pi^2 kt/1600} + \frac{1}{3}\sin(3\pi x/40)e^{-9\pi^2 kt/1600} + \frac{1}{5}\sin(5\pi x/40)e^{-25\pi^2 kt/1600} + \dots\right).$$

Since the rod is made of copper, we see from the table on page 622 that the value of the thermal diffusivity k is 1.15 cm^2/s. Since $k/1600 = (1.15)/1600 = 0.00071875$, we find that at $t = 5$ min, or $t = 300$ seconds, and $x = 20$, the first term in the series for $u(x,t)$ has the value

$$\frac{400}{\pi}\sin\left(\pi\frac{20}{40}\right)e^{-\pi^2(0.0007)(300)} = (127.3) \times 1 \times e^{-2.128} \approx 15.2.$$

The second term in the series for $u(x,t)$ has the value

$$\frac{400}{3\pi}\sin\left(3\pi\frac{20}{40}\right)e^{-9\pi^2(0.0007)(300)} = (42.4) \times -1 \times e^{-19.2},$$

and since $e^{-19} \approx 6 \times 10^{-9}$, this term is negligible. The third and succeeding terms are even smaller, so we can assume that u is approximately equal to the first term. Hence $u \approx 15.2^\circ$ C.

(c) Since the rod is made of concrete, we see from the table on page 622 that the value of k is 0.005 cm^2/s. So $k/1600 = 3.125 \times 10^{-6}$, and at $x = 20$ the first term in the series for $u(x,t)$ is

$$\frac{400}{\pi}\sin\left(\pi\frac{20}{40}\right)e^{-\pi^2 t(3.125\times 10^{-6})} = 127.3\ e^{(-3.1\times 10^{-5})t}.$$

Setting equal to 15 and solving for t, we find that

$$t = \frac{-\ln\left(\frac{15}{127.3}\right)}{3.1 \times 10^{-5}} = 0.69 \times 10^5 \text{ seconds} = 19.2 \text{ hours}.$$