EXAM 2
Math 3413
10-~~28~~-13 25

Name key

Instructions *Work all of the following problems in the space provided. If there is not enough room, you may write on the back sides of the pages. Give thorough explanations to receive full credit.*

1. *(18 points)* Consider the inhomogeneous linear differential equation

$$y'' - 5y' + 6y = 4e^{3x}.$$

a) Find the general solution y_c of the associated homogeneous equation.

[6] (2) $r^2 - 5r + 6 = 0$

$(r-2)(r-3) = 0$

(2) $r = 2$ or $r = 3$

$\rightarrow \boxed{y_c = C_1 e^{2x} + C_2 e^{3x}}$ (2)

b) Use the method of undetermined coefficients to find a particular solution y_p of the inhomogeneous equation.

[9] e^{3x} duplicates a term in y_c, so use xe^{3x} instead. (2)

If $y = Axe^{3x}$, then $y' = Ae^{3x} + 3Axe^{3x}$ (2)

and $y'' = 3Ae^{3x} + (3Ae^{3x} + 9Axe^{3x}) = 6Ae^{3x} + 9Axe^{3x}$ (2)

So $y'' - 5y' + 6y = 4e^{3x} \Rightarrow (6Ae^{3x} + 9Axe^{3x}) - 5(Ae^{3x} + 3Axe^{3x}) + 6Axe^{3x} = 4e^{3x}$

Collecting terms we get $(6A - 5A)e^{3x} + (9A - 15A + 6A)xe^{3x} = 4e^{3x}$

or $Ae^{3x} + 0 = 4e^{3x}$. (1) So $A = 4$ (1) and $\boxed{y_p = 4xe^{3x}}$ (1)

c) Find the general solution of the inhomogeneous equation.

[3] It is $y = y_c + y_p$, or $\boxed{y = C_1 e^{2x} + C_2 e^{3x} + 4xe^{3x}}$

2. *(12 points)* Consider the differential equation

$$\frac{d^4y}{dx^4} - 9\frac{d^2y}{dx^2} = 0.$$

a) Find the characteristic equation and its roots.

[6] $r^4 - 9r^2 = 0$ (2) $\Rightarrow r^2(r^2 - 9) = 0$ (2) $\Rightarrow$ $r = 0$ double root, $r = 3$, $r = -3$ (2)

b) Give the general solution of the differential equation.

[6] (2)

$y = Ae^{0x}$ (2) $+ Bxe^{0x}$ (2) $+ Ce^{3x} + De^{-3x}$

or $\boxed{y = A + Bx + Ce^{3x} + De^{-3x}}$

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3. *(18 points)* Consider the equation

$$y'' - 4y = 8x^2$$

a) Find the general solution y_c of the associated homogeneous equation.

[4] $r^2 - 4 = 0 \Rightarrow r = 2$ and $r = -2$, so

$$y_c = C_1 e^{2x} + C_2 e^{-2x}$$

[8] *b)* Use the method of undetermined coefficients to find a particular solution y_p of the inhomogeneous equation.

The ~~xxxx~~ inhomogeneous term is x^2 and its derivatives are $2x$ and 1, so use $y = A + Bx + Cx^2$. 2

Then $\begin{cases} y' = B + 2Cx \\ y'' = 2C \end{cases}$ and $y'' - 4y = 8x^2 \Rightarrow$

$$\Rightarrow 2C - 4(A + Bx + Cx^2) = 8x^2 \quad 3$$

$$\Rightarrow (2C - 4A) - 4Bx - 4Cx^2 = 8x^2$$

So $\begin{cases} 2C - 4A = 0 \\ 4B = 0 \\ -4C = 8 \end{cases}$, which gives $2A = C$, $B = 0$, and $C = -2$ 3

So $A = -1$, and

$$y_p = -1 + 0x - 2x^2 = \boxed{-2x^2 - 1}$$

[6] *c)* Find a solution of the inhomogeneous equation which satisfies the conditions $y(0) = 0$ and $y'(0) = 2$. 1

The general solution is $y = y_c + y_p = Pe^{2x} + Qe^{-2x} - 2x^2 - 1$.

Then $y(0) = 0$ implies $Pe^0 + Qe^0 - 0 - 1 = 0 \Rightarrow P + Q - 1 = 0$

$\Rightarrow P + Q = 1$ 1

and $y'(x) = 2Pe^{2x} - 2Qe^{-2x} - 4x$, 1

so $y'(0) = 2 \Rightarrow 2Pe^0 - 2Qe^0 - 0 = 2 \Rightarrow 2P - 2Q = 2$

$\Rightarrow P - Q = 1$ 1

Since $\begin{matrix} P + Q = 1 \\ P - Q = 1 \end{matrix}$, solving gives $P = 1$ and $Q = 0$. So 1

$$y = 1 \cdot e^{2x} + 0e^{-2x} - 2x^2 - 1$$

$$= \boxed{e^{2x} - 2x^2 - 1} \quad 1$$

4. *(20 points)* In this question you are asked to use Laplace transforms to solve the initial-value problem

$$x'' + 3x' + 2x = 0, \quad x(0) = 2, \quad x'(0) = 1.$$

[10]

a) Find an equation for the Laplace transform $X(s)$ of $x(t)$. 2

Taking $\mathcal{L}$ of both sides gives 2

$$[s^2 X(s) - s\,x(0) - x'(0)] + 3[s X(s) - x(0)] + 2X(s) = 0.$$

So $s^2 X(s) - 2s - 1$ 2 $+ 3[sX(s) - 2] + 2X(s) = 0$

or $(s^2+3s+2)X(s) - 2s - 7 = 0$ 2

b) Solve the equation in part *a)* to find $X(s)$.

We get $(s^2+3s+2)X(s) = 2s+7$, 2

so $X(s) = \dfrac{2s+7}{(s^2+3s+2)}$.

[10]

c) Use your answer to part *b)* to find $x(t)$. (Hint: do a partial fraction decomposition.)

$X(s) = \dfrac{2s+7}{(s+2)(s+1)} = \dfrac{A}{s+1} + \dfrac{B}{s+2}$ 2, where $2s+7 = A(s+2) + B(s+1)$. 2

Putting $s=-2$ gives $3 = B(-1)$, so $B = -3$

" $s=-1$ " $5 = A$. So $X(s) = \dfrac{5}{s+1} - \dfrac{3}{s+2}$. 2

Then $x(t) = \mathcal{L}^{-1}\left\{\dfrac{5}{s+1}\right\} - \mathcal{L}^{-1}\left\{\dfrac{3}{s+2}\right\}$ 2

$= \boxed{5e^{-t} - 3e^{-2t}}$ 2

5. *(12 points)* Find $\mathcal{L}\{e^{2t}t\cos(3t)\}$, showing all work.

Put $f(t) = t\cos 3t$, then $\mathcal{L}\{f\} = F(s) = -\dfrac{d}{ds}\mathcal{L}\{\cos 3t\}$ 2

$= -\dfrac{d}{ds}\left\{\dfrac{s}{s^2+9}\right\}$ 2

$= -\left\{\dfrac{(s^2+9)\cdot 1 - s(2s)}{(s^2+9)^2}\right\}$ 2

$= -\left\{\dfrac{s^2+9-2s^2}{(s^2+9)^2}\right\} = \dfrac{s^2-9}{(s^2+9)^2}$.

2

So $\mathcal{L}\{e^{2t}f(t)\}$ 2

$= F(s-2)$

$= \boxed{\dfrac{(s-2)^2-9}{[(s-2)^2+9]^2}}$ 2

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6. *(20 points)* In this question you are asked to use Laplace transforms to solve the initial-value problem

$$x' + 3x = 25te^{2t}, \quad x(0) = 0.$$

[8]

a) Find an equation for the Laplace transform $X(s)$ of $x(t)$.

Taking $\mathcal{L}$ of both sides gives (2)

$$[sX(s) - x(0)] + 3X(s) = 25\mathcal{L}\{te^{2t}\} \quad (2)$$

or $$sX(s) - 0 + 3X(s) = 25\left(-\frac{d}{ds}\mathcal{L}\{e^{2t}\}\right) \quad (2)$$

or $$(s+3)X(s) = -25\frac{d}{ds}\left(\frac{1}{s-2}\right) = -25\left(-(s-2)^{-2}\right) = \frac{25}{(s-2)^2}$$

b) Solve the equation in part *a)* to find $X(s)$.

$$(s+3)X(s) = \frac{25}{(s-2)^2} \Rightarrow X(s) = \frac{25}{(s+3)(s-2)^2} \quad (2)$$

[12]

c) Use your answer to part *b)* to find $x(t)$. (Hint: do a partial fraction decomposition.)

$$X(s) = \frac{25}{(s+3)(s-2)^2} = \frac{A}{s+3} + \frac{B}{s-2} + \frac{C}{(s-2)^2}. \quad (2)$$

where $$25 = A(s-2)^2 + B(s-2)(s+3) + C(s+3) \quad (2)$$

Putting $s=2$: $25 = 0 + 0 + 5C \Rightarrow \boxed{C=5}$ (1)

Putting $s=-3$: $25 = A(-5)^2 + 0 + 0 \Rightarrow 25 = 25A$
$\Rightarrow \boxed{A=1}$ (1)

Putting $s=0$: $25 = A\cdot(-2)^2 + B(-2)(3) + C\cdot 3$

$\Rightarrow 25 = 4A - 6B + 3C = 4 - 6B + 15$

$\Rightarrow 25 = 19 - 6B$

$\Rightarrow 6 = -6B \Rightarrow \boxed{B=-1}$ (1)

So $$X(s) = \frac{1}{s+3} - \frac{1}{s-2} + \frac{5}{(s-2)^2}. \quad (1)$$

and $$x(t) = \mathcal{L}^{-1}\left\{\frac{1}{s+3}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} + 5\mathcal{L}^{-1}\left\{\frac{1}{(s-2)^2}\right\}$$

or $$x(t) = e^{-3t} \ (1) - e^{2t} \ (1) + 5\mathcal{L}^{-1}\{F(s-2)\}$$ where $F(s) = \frac{1}{s^2}$, $f(t) = t$

$$x(t) = e^{-3t} - e^{2t} + 5\cdot e^{2t}f(t) \quad (1)$$

$$= \boxed{e^{-3t} - e^{2t} + 5te^{2t}} \quad (1)$$