EXAM 1 Name ~~key~~
Math 3413
9-20-13

Instructions *Work all of the following problems in the space provided. If there is not enough room, you may write on the back sides of the pages. Give thorough explanations to receive full credit.*

1. *(10 points)* **Determine whether $y = x^3$ is a solution of the equation**

$$x^2\frac{d^3y}{dx^3} - 3\frac{dy}{dx} + \frac{2}{x}y = 0.$$

Show all work.

③
$$y = x^3$$
$$\frac{dy}{dx} = 3x^2$$
$$\frac{d^2y}{dx^2} = 6x$$
$$\frac{d^3y}{dx^3} = 6$$

Substituting into the left-hand side of the equation gives

④ $$x^2 \cdot 6 - 3\cdot 3x^2 + \frac{2}{x}\cdot x^3 = 6x^2 - 9x^2 + 2x^2 = -x^2$$

③ which is not the same function as 0.
So $\boxed{y = x^3 \text{ is not a solution to the equation}}$

2. *(15 points)* **Consider the linear equation**

$$\frac{dy}{dx} + (\cos x)y = e^{-\sin x}$$

a) **Find the general solution.**

[11] The integrating factor is $e^{\int \cos x\, dx} = e^{\sin x}$. ②
Multiplying both sides of the equation by $e^{\sin x}$, we get ②

$$e^{\sin x}\frac{dy}{dx} + e^{\sin x}\cos x \cdot y = e^{\sin x}e^{-\sin x} = e^0 = 1.$$ ②

Or:
$$\frac{d}{dx}\left(e^{\sin x}\cdot y\right) = 1.$$

Integrating gives $e^{\sin x}\cdot y = x + C$ ② ②

Dividing by $e^{\sin x}$ gives $\boxed{y = \left[\frac{x+C}{e^{\sin x}}\right]}$. ①

b) **Find the particular solution which satisfies the condition $y(0) = 4$.**

[4] ② Putting $y=4$ and $x=0$ in the general solution gives:

$$4 = \frac{0+C}{e^{\sin 0}} = \frac{C}{e^0} = \frac{C}{1} = C$$

So $C=4$, and $\boxed{y = \left[\frac{x+4}{e^{\sin x}}\right]}$ ②

 Name ______________________

3. *(18 points)* **Find the general solution of the equation**

$$\frac{dy}{dx} = -\frac{x}{2y} + \frac{3y}{2x}.$$

(Hint: the equation is homogeneous. Put $y = vx$ and reduce to a separable equation for v.)

If $y = vx$, then

(2) $\frac{dy}{dx} = v \cdot 1 + \frac{dv}{dx} \cdot x$, so substituting into the equation

gives $v + \frac{dv}{dx} \cdot x = \frac{-x}{2vx} + \frac{3vx}{2x}$ (2)

or $v + \frac{dv}{dx} \cdot x = \frac{-1}{2v} + \frac{3v}{2}$ (2)

So $\frac{dv}{dx} \cdot x = \frac{-1}{2v} + \frac{3v}{2} - v = \frac{-1}{2v} + \frac{v}{2} = \frac{-1+v^2}{2v}.$

Separating variables gives

$$\frac{dv}{dx} \cdot x = \frac{v^2-1}{2v} \quad (2)$$

$$\Rightarrow \int \frac{2v}{v^2-1} \cdot dv = \int \frac{dx}{x} \quad (2)$$

$\Rightarrow$ $\left(\begin{matrix} u = v^2-1 \\ du = 2v \end{matrix} \right)$

$$\int \frac{du}{u} = \int \frac{dx}{x}$$

$\Rightarrow \ln u = \ln x + C$ (2) $\Rightarrow e^{\ln u} = e^{\ln x + C} = e^{\ln x} \cdot e^C$

$\Rightarrow u = x \cdot D$ (2)

So $v^2 - 1 = Dx \Rightarrow v^2 = Dx + 1$

$\Rightarrow v = \pm\sqrt{Dx+1}$ (2)

$\Rightarrow \frac{y}{x} = \pm\sqrt{Dx+1}$

$\Rightarrow y = \pm x\sqrt{Dx+1}$ (2)

4. *(15 points)* **For the exact differential equation,**

$$(3x^2 + 4xy) + (2x^2 + 2y)\frac{dy}{dx} = 0,$$

find the general solution by giving an implicit equation for y.

The general solution is $F(x,y) = C$ where $\frac{\partial F}{\partial x} = 3x^2 + 4xy$ 2
and $\frac{\partial F}{\partial y} = 2x^2 + 2y$

Then $F = \int^x (3x^2 + 4xy) = x^3 + 2x^2y + g(y)$, 2 2

so $\frac{\partial F}{\partial y} = 0 + 2x^2 + g'(y) = 2x^2 + 2y$ 2, which implies $g'(y) = 2y$, 2

or $g(y) = y^2 + D$. 2 So $F = x^3 + 2x^2y + y^2 + D$ 1, and y is

given implicitly by $x^3 + 2x^2y + y^2 + D = C$, or $\boxed{x^3 + 2x^2y + y^2 = E}$ 2

5. *(15 points)* **Find the general solution of**

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = \frac{1}{x}.$$

(Hint: Since y does not appear in the equation, you should substitute $p = dy/dx$.)

If $p = \frac{dy}{dx}$, then $\frac{dp}{dx} = \frac{d^2y}{dx^2}$ 2, so the equation can be rewritten as

$x^2\frac{dp}{dx} + xp = \frac{1}{x}$, 2

or $\frac{dp}{dx} + \frac{1}{x}\cdot p = \frac{1}{x^3}$. 1 This is a linear equation, with integrating factor $e^{\int \frac{1}{x}dx} = e^{\ln x} = x$. 2

Multiplying both sides by x gives

$x \cdot \frac{dp}{dx} + p = \frac{1}{x^2}$ 2, or $\frac{d}{dx}(xp) = \frac{1}{x^2}$. Integrating

both sides gives $xp = \int \frac{1}{x^2}dx = -\frac{1}{x} + C$, 2

and dividing by x gives $p = -\frac{1}{x^2} + \frac{C}{x}$. 1

So $\frac{dy}{dx} = \frac{-1}{x^2} + \frac{C}{x}$. 1 Integrating again gives $\boxed{y = \frac{1}{x} + C\ln|x| + D}$ 2

6. *(17 points)* **Find the solution of the equation**

$$\frac{d^2y}{dx^2} - 7\,\frac{dy}{dx} + 10y = 0$$

which satisfies the initial conditions $y(0) = 2$ and $y'(0) = 1$.

Putting $y = e^{rx}$ into the equation gives

$$r^2 e^{rx} - 7re^{rx} + 10e^{rx} = 0, \quad 2$$

or $r^2 - 7r + 10 = 0$ 2, so $(r-2)(r-5) = 0$ 2

and $r = 2$ or $r = 5$. So two independent solutions are $y_1 = e^{2x}$ and $y_2 = e^{5x}$, and the general solution is

$$y = Ae^{2x} + Be^{5x}. \quad 2$$

Then $\frac{dy}{dx} = 2Ae^{2x} + 5Be^{5x}$ 2. Putting $x = 0$ and using the given initial conditions, we get

$$2 = A + B \quad 2$$
$$1 = 2A + 5B \quad 2$$

Multiplying the first equation by 2 and subtracting it from the second gives $-3 = 3B$, or $B = -1$, and so $A = 3$. 2 So $\boxed{y = 3e^{2x} - e^{5x}.}$ 1

7. *(10 points)* **Find the general solution of the equation**

$$\frac{d^2y}{dx^2} - 8\,\frac{dy}{dx} + 16y = 0.$$

Putting $y = e^{rx}$ into the equation gives

$$r^2 e^{rx} - 8re^{rx} + 16e^{rx} = 0, \quad 2$$
$$r^2 - 8r + 16 = 0, \quad 2$$
$$(r-4)^2 = 0, \quad 2$$

which has $r = 4$ as a double root.

So two independent solutions of the equation are

$$y_1 = e^{4x} \text{ and } y_2 = xe^{4x}. \quad 2$$

The general solution is 2

$$\boxed{y = Ae^{4x} + Bxe^{4x}.}$$