

Instructions Work all of the following problems in the space provided. If there is not enough room, you may write on the back sides of the pages. Give thorough explanations to receive full credit.

1. (15 points) For the function $f(x, y) = \sin(xy^2)$, find:

[3] a) $f_x = \boxed{\cos(xy^2) \cdot y^2}$

[3] b) $f_{xx} = -\sin(xy^2) \cdot y^2 \cdot y^2 = \boxed{-\sin(xy^2) y^4}$

[3] c) $f_{xy} = -\sin(xy^2) \cdot x \cdot 2y \cdot y^2 + \cos(xy^2) \cdot 2y = \boxed{-2xy^3 \sin(xy^2) + 2y \cos(xy^2)}$

[3] d) $f_y = \boxed{\cos(xy^2) \cdot x \cdot 2y}$

[3] e) $f_{yx} = -\sin(xy^2) \cdot y^2 \cdot x \cdot 2y + \cos(xy^2) \cdot 2y = \boxed{\text{same as } f_{xy}}$

2. (15 points) The surface S is given by the equation $z = \sqrt{x+y^2}$.

a) Find the equation of the tangent plane to S at the point P on the surface where $x = 5$ and $y = 2$.

[10] $\frac{\partial z}{\partial x} \Big|_P = \frac{1}{2} (x+y^2)^{-1/2} \Big|_{\substack{x=5 \\ y=2}} = \frac{1}{2} (5+4)^{-1/2} = \frac{1}{2\sqrt{9}} = \frac{1}{6}$, (3)

$\frac{\partial z}{\partial y} \Big|_P = \frac{1}{2} (x+y^2)^{-1/2} \cdot 2y \Big|_{\substack{x=5 \\ y=2}} = \frac{1}{2} (5+4)^{-1/2} \cdot 4 = \frac{1}{6} \cdot 4 = \frac{2}{3}$, (3)

and $z \Big|_P = \sqrt{5+4} = 3$. So equation is $\boxed{z - 3 = \frac{1}{6}(x-5) + \frac{2}{3}(y-2)}$ (4)

b) Use the tangent plane to find an approximate value of z when $x = 5.06$ and $y = 2.09$.

[5] Putting $x = 5.06$ and $y = 2.09$ into the equation gives

$z - 3 = \frac{1}{6}(0.06) + \frac{2}{3}(0.09) = 0.01 + 0.06 = 0.07$,

So $\boxed{z = 3.07}$ (is an approximation to $\sqrt{5.06 + (2.09)^2}$)

3. (20 points) The surface S is given by the equation $F(x, y, z) = 7$, where $F(x, y, z) = z^3 + xy^2z^2 - xyz$.

[5] a) Find $\frac{\partial z}{\partial x} = \frac{-F_x}{F_z} = \frac{-(y^2z^2 - yz)}{(3z^2 + xy^2 \cdot 2z - xy)}$ (3)

[5] b) Find $\frac{\partial z}{\partial y} = \frac{-F_y}{F_z} = \frac{-(2xyz^2 - xz)}{(3z^2 + xy^2 \cdot 2z - xy)}$ (3)

[5] c) Find $\nabla F = F_x \vec{i} + F_y \vec{j} + F_z \vec{k} = \langle y^2z^2 - yz, 2xyz^2 - xz, 3z^2 + xy^2 \cdot 2z - xy \rangle$ (2)

[5] d) Find the equation of the tangent plane to S at the point $P(3, 2, 1)$.

At P we have

$$\vec{\nabla} F|_P = \langle 4 \cdot 1 - 2 \cdot 1, 2 \cdot 3 \cdot 2^2 - 3 \cdot 1, 3 \cdot 1 + 3 \cdot 4 \cdot 2 - 3 \cdot 2 \rangle = \langle 2, 9, 21 \rangle$$

So the equation is $\vec{\nabla} F|_P \cdot \langle x-3, y-2, z-1 \rangle = 0$ or $2(x-3) + 9(y-2) + 21(z-1) = 0$ (3)

4. (20 points) The function f is given by $f(x, y) = xe^{2y} + x^2 \cos y$.

[6] a) Find $\nabla f = f_x \vec{i} + f_y \vec{j} = \langle e^{2y} + 2x \cos y, x \cdot 2e^{2y} + x^2 \cdot (-\sin y) \rangle$ (2)

[6] b) Find the directional derivative of f at the point $P(1, 0)$ in the direction of the vector $\langle 1, 5 \rangle$.

A unit vector in this direction is $\vec{u} = \frac{\langle 1, 5 \rangle}{\sqrt{26}}$, and

$$D_{\vec{u}} f|_P = \vec{\nabla} f|_P \cdot \vec{u} = \langle e^0 + 2 \cdot 1 \cdot \cos 0, 1 \cdot 2 \cdot e^0 + 0 \rangle \cdot \frac{\langle 1, 5 \rangle}{\sqrt{26}} = \frac{\langle 3, 2 \rangle \cdot \langle 1, 5 \rangle}{\sqrt{26}} = \frac{13}{\sqrt{26}}$$

c) At the point $P(1, 0)$, determine the unit vector \vec{u} in the direction of which f has the maximum rate of change.

[4] The maximum rate of change is in the direction of the gradient vector $\vec{\nabla} f|_P = \langle 3, 2 \rangle$, and the unit vector in this direction is $\vec{u} = \frac{\langle 3, 2 \rangle}{\sqrt{9+4}} = \frac{\langle 3, 2 \rangle}{\sqrt{13}}$ (2)

d) Find the directional derivative of f in the direction of the vector \vec{u} found in part c).

[4] For $\vec{u} = \frac{\langle 3, 2 \rangle}{\sqrt{13}}$, $D_{\vec{u}} f|_P = \langle 3, 2 \rangle \cdot \frac{\langle 3, 2 \rangle}{\sqrt{13}} = \frac{13}{\sqrt{13}} = \sqrt{13}$ (1)

5. (15 points) Find the local maximum and minimum values and saddle points of the function

$$f(x, y) = 6x^2 - 2x^3 + 3y^2 + 6xy.$$

(Remember that a critical point is a saddle point if $D = f_{xx}f_{yy} - f_{xy}^2 < 0$, is a local maximum if $D > 0$ and $f_{xx} < 0$, and is a local minimum if $D > 0$ and $f_{xx} > 0$.)

Critical points are where

$$0 = f_x = 12x - 6x^2 + 6y \quad (1)$$

$$\text{and } 0 = f_y = 6y + 6x. \quad (2)$$

So $6y + 6x = 0 \Rightarrow y = -x$ (1), and substituting ~~the~~ ~~in~~ in the first equation gives

$$12x - 6x^2 - 6x = 0 \Rightarrow 6x - 6x^2 = 0 \quad (1)$$

$$\Rightarrow x(1-x) = 0$$

$$\Rightarrow x = 0 \text{ or } x = 1. \quad (2)$$

When $x = 0$, $y = -x = 0$, so $(0, 0)$ is a critical point. (1)

When $x = 1$, $y = -x = -1$, so $(1, -1)$ is a critical point. (1)

At $(0, 0)$, $f_{xx} = 12 - 12x = 12$, so $D = 72 - 36 > 0$ (2)
 $f_{yy} = 6$
 $f_{xy} = 6$
and $f_{xx} > 0$,
so $(0, 0)$ is a local minimum. (1)

At $(1, -1)$, $f_{xx} = 12 - 12 = 0$, so $D = 0 - 36 < 0$ (2)
 $f_{yy} = 6$
 $f_{xy} = 6$
so $(1, -1)$ is a saddle point. (1)

6. (15 points) Evaluate the iterated integral $\int_0^1 \int_0^1 \sqrt{1+2x+3y} \, dx \, dy$, showing all work.

$$= \int_0^1 \left[\int_{1+3y}^{3+3y} \sqrt{u} \frac{du}{2} \right] dy \quad \left(\begin{array}{l} u = 1+2x+3y \quad (1) \\ du = 2dx \\ \frac{du}{2} = dx \quad (1) \end{array} \quad \begin{array}{l} x=0 \rightarrow u=1+3y \\ x=1 \rightarrow u=3+3y \end{array} \right)$$

$$= \int_0^1 \frac{1}{2} \left[\frac{2}{3} u^{3/2} \right]_{u=1+3y}^{u=3+3y} dy$$

$$= \int_0^1 \frac{1}{3} \left[(3+3y)^{3/2} - (1+3y)^{3/2} \right] dy \quad (2)$$

$$= \frac{1}{3} \int_0^1 (3+3y)^{3/2} dy - \frac{2}{3} \int_0^1 (1+3y)^{3/2} dy$$

$$\left(\begin{array}{l} w = 3+3y \\ dw = 3dy \\ \frac{dw}{3} = dy \end{array} \quad \begin{array}{l} y=0 \rightarrow w=3 \\ y=1 \rightarrow w=6 \end{array} \right)$$

$$\left(\begin{array}{l} w = 1+3y \\ dw = 3dy \\ \frac{dw}{3} = dy \end{array} \quad \begin{array}{l} y=0 \rightarrow w=1 \\ y=1 \rightarrow w=4 \end{array} \right)$$

$$= \frac{1}{3} \int_3^6 w^{3/2} \frac{dw}{3} - \frac{2}{3} \int_1^4 w^{3/2} \frac{dw}{3} \quad (5)$$

$$= \frac{1}{9} \int_3^6 w^{3/2} dw - \frac{2}{9} \int_1^4 w^{3/2} dw$$

$$= \frac{1}{9} \cdot \frac{2}{5} \left[w^{5/2} \right]_3^6 - \frac{2}{9} \cdot \frac{2}{5} \left[w^{5/2} \right]_1^4 \quad (5)$$

$$= \left[\frac{2}{45} \left[6^{5/2} - 3^{5/2} \right] - \frac{2}{45} \left[4^{5/2} - 1^{5/2} \right] \right]$$

(This could be simplified a bit $\frac{2}{45} [6^{5/2} - 3^{5/2} - 32 + 1]$)