

## Trigonometry Review

A student asked what trigonometric facts and identities should be memorized. I've listed the essentials below, selected from Appendix D of our text.

**Radian measure of angles.** The formulas that you've learned for derivatives and integrals of trigonometric functions are only correct when the functions are defined using radian measures of angles. (If we define a function  $f(x)$  by setting  $f(x)$  equal to the sine of the angle whose measure is  $x$  degrees, then the derivative of this function is *not* the cosine of the angle whose measure is  $x$  degrees.) So in this course it's safest to express all angles in radian measure.

If you prefer can still use degrees to measure angles sometimes, but only if you're sure that by doing so you don't mess up a derivative or integral formula. When you want to convert from degrees to radians, remember that 360 degrees is  $2\pi$  radians — so you have to multiply degrees by  $2\pi/360$ , or  $\pi/180$ , to get radians.

**Definition and values of trigonometric functions.** For angles  $\theta$  between 0 and 90 degrees, you can define the trigonometric functions by drawing a right triangle with angle  $\theta$  in it, and letting  $\sin \theta$  be the opposite side over the hypotenuse and  $\cos \theta$  be the adjacent side over the hypotenuse (see figure 6 on page A26 in Appendix D). Then you can figure out the values of the other four trig functions by using the facts that

$$(1) \quad \tan \theta = \frac{\sin \theta}{\cos \theta}, \quad \sec \theta = \frac{1}{\cos \theta}, \quad \csc \theta = \frac{1}{\sin \theta}.$$

You should memorize the side lengths of the 45-45-90 triangles and 30-60-90 triangles (Figure 8 on page A26). With those memorized you then automatically know the values of all the trigonometric functions at  $\theta = \pi/6$ ,  $\pi/4$ , and  $\pi/3$ .

To find the sines and cosines of angles  $\theta$  of more than 90 degrees, or less than 0 degrees, you have to use the definitions

$$(2) \quad \cos \theta = \frac{x}{r}, \quad \sin \theta = \frac{y}{r}.$$

Here  $x$  and  $y$  are the  $x$  and  $y$  coordinates of a point  $P$  which lies on the line through the origin which makes an angle  $\theta$  with the positive  $x$ -axis; and  $r$  is the distance from  $P$  to the origin. (It's often easiest to take  $r = 1$ , so  $P$  is on the unit circle.) Once you know the sine and cosine, you can find the values of the other trigonometric functions using the formulas in equations (1) above. The procedure of how to use these formulas is illustrated in Example 3 on page A27.

You can use the above definition and your knowledge of the 45-45-90 and 30-60-90 triangles to easily figure out the sine and cosine of any angle which is a multiple of 30 or 45 degrees (including negative angles). You don't need to memorize tables like the one in blue on page A27.

Quite often you have to find the angles at which the sine and cosine take the values 0, 1, and  $-1$ . The easiest way to do this is by using the formulas in (2). For example, to find the values of  $\theta$  for which  $\cos \theta = -1$ , you think of the point  $P$  on the unit circle whose  $x$ -coordinate is  $-1$ ; that is, the point  $P = (-1, 0)$ . This is on a line which makes an angle of 180 degrees, or  $\pi$  radians, with the positive  $x$ -axis, so the cosine of  $\pi$  is zero. But the same line also makes other angles with the positive  $x$ -axis: namely,  $\pi \pm 2\pi$ ,  $\pi \pm 4\pi$ , etc. So the angles whose cosine are zero are:  $\pi$  plus or minus any multiple of  $2\pi$ . In other words,  $\cos \theta = 0$  if  $\theta$  is in the set

$$\{\dots, -7\pi, -5\pi, -3\pi, -\pi, \pi, 3\pi, 5\pi, 7\pi, \dots\}.$$

**Trigonometric identities.** You should know the basic trigonometric identity,

$$(3) \quad \sin^2 \theta + \cos^2 \theta = 1.$$

Another identity that comes up a lot is

$$(4) \quad \tan^2 \theta + 1 = \sec^2 \theta,$$

but one bit of good news is that you don't have to remember (4) separately, since you can always derive it from (3) when you need it, simply by dividing (3) by  $\cos^2 \theta$  and using the first two definitions in (1).

There are a number of other trigonometric identities in Appendix D, all of which are occasionally useful, but the only other ones I recommend you memorize are:

$$\sin(-\theta) = -\sin \theta, \quad \cos(-\theta) = \cos(\theta)$$

and

$$\begin{aligned} \sin(2\theta) &= 2 \sin \theta \cos \theta \\ \cos(2\theta) &= \cos^2 \theta - \sin^2 \theta. \end{aligned}$$

Any other identities you might need on the exams, I'll supply to you on the exams themselves.

**Graphs of the trigonometric functions.** You should memorize the graphs of the trigonometric functions (Figure 13 of Appendix D).

That completes the review of what you need to remember of trigonometry from pre-calculus. As an added bonus, here's a summary of what you will need to remember of trigonometry from calculus:

**Derivatives and antiderivatives of trigonometric functions.** You should memorize the formulas for the derivatives of the six trigonometric functions:

$$\begin{aligned} \frac{d}{dx}(\sin x) &= \cos x & \frac{d}{dx}(\csc x) &= -\csc x \cot x \\ \frac{d}{dx}(\cos x) &= -\sin x & \frac{d}{dx}(\sec x) &= \sec x \tan x \\ \frac{d}{dx}(\tan x) &= \sec^2 x & \frac{d}{dx}(\cot x) &= -\csc^2 x \end{aligned}$$

Another bit of good news is that you don't have to remember the corresponding antiderivative formulas separately — they're just the reverse of the above formulas. For example,

$$\text{Since } \frac{d}{dx}(\tan x) = \sec^2 x, \text{ then } \int \sec^2 x \, dx = \tan x + C.$$

A couple of integrals of basic trig functions that do not come as the reverse of one of the above formulas are:

$$\int \tan x \, dx = \ln |\sec x| + C, \quad \int \sec x \, dx = \ln |\sec x + \tan x| + C.$$

The integral of  $\tan x$  you can derive for yourself by writing  $\int \tan x \, dx = \int \frac{\sin x}{\cos x} \, dx$  and making the substitution  $u = \cos x$ . The integral of  $\sec x$  is considerably harder to derive (I've talked about it in class a couple of times). Anyway, it's probably worth memorizing both of these, just to have them handy if you need them.