Quiz 6

Name: key

Determine whether each of the following series is absolutely convergent, conditionally convergent, or divergent. Give reasons for your answers.

[7] **1.** $\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2+1}$

Let $b_n = \frac{n}{n^2+1}$. Then $b_n \geq 0$ for all $n = 1, 2, 3, \ldots$; and b_n is decreasing*, and $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{n}{n^2+1} = \lim_{n\to\infty} \frac{1/n}{1+\frac{1}{n^2}} = \frac{0}{1+0} = 0$.

So $\sum (-1)^n b^n$ converges by the Alternating series test. (3)

But $\sum \frac{n}{n^2+1}$ diverges by the limit comparison test: since $\sum \frac{1}{n}$ diverges and $\lim_{n\to\infty} \frac{\left(\frac{n}{n^2+1}\right)}{\left(\frac{1}{n}\right)} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = \lim_{n\to\infty} \frac{1}{1+\frac{1}{n^2}} = \frac{1}{1+0} = 1 \neq 0$. (3)

So the series is conditionally convergent. (1)

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[6] **2.** $\sum_{n=1}^{\infty} \frac{(-1)^n n^2}{n^2+1}$

Since $\lim_{n\to\infty} \frac{n^2}{n^2+1} = 1$, Then $\frac{(-1)^n n^2}{n^2+1}$ does not have a limit of 0 (it bounces back and forth between numbers getting closer to 1 and numbers getting closer to (-1).) (3)

So the series diverges, by the Theorem which says that a series $\sum a_n$ can only converge if $\lim a_n = 0$. (3)

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[7] **3.** $\sum_{n=1}^{\infty} \frac{2^n \cos n}{n!}$

Here $a_n = \frac{2^n \cos n}{n!}$, so $|a_n| \leq \frac{2^n}{n!}$.

But $\sum \frac{2^n}{n!}$ converges by the Ratio Test, since (4)

$$\lim_{n\to\infty} \frac{\left(\frac{2^{n+1}}{(n+1)!}\right)}{\left(\frac{2^n}{n!}\right)} = \lim_{n\to\infty} \frac{2^{n+1}}{2^n} \frac{n!}{(n+1)!} = \lim_{n\to\infty} 2 \cdot \frac{1}{n+1} = 0 < 1.$$

(2)

So $\sum |a_n|$ converges by the comparison test. (1)

Therefore $\sum a_n$ converges absolutely.

* You can prove this by taking the derivative of $f(x) = \frac{x}{x^2+1}$ and showing that the derivative is less than 0.