EXAM 3
Math 2924
4-24-15

Name key

Instructions *Work all of the following problems in the space provided. If there is not enough room, you may write on the back sides of the pages. Give thorough explanations to receive full credit.*

1. *(10 points)* Find the sum of the infinite geometric series

$$2 + 0.2 + 0.02 + 0.002 + 0.0002 + \dots$$

3

We know that if $|x|<1$, then $a + ax + ax^2 + ax^3 + \dots = \frac{a}{1-x}$.

2 2

In this case, $a = 2$ and $x = \frac{1}{10}$, so the sum is

$$\frac{a}{1-x} = \frac{2}{1-\frac{1}{10}} = \frac{2}{(9/10)} = \boxed{\frac{20}{9}}.$$ 3

2. *(15 points)* Decide, giving a valid reason, whether the series

$$\sum_{n=1}^{\infty} \frac{n^3 + 20n^2 + 100}{5n^7 - n - 10}$$

converges. (Hint: use the limit comparison test.)

Let $a_n = \frac{n^3+20n^2+100}{5n^7-n-10}$ and $b_n = \frac{1}{n^4}$. (For n large enough, $a_n \geq 0$ and $b_n \geq 0$.)

Then $\sum b_n = \sum \frac{1}{n^4}$ converges (it is a p-series with $p>1$). 3

And $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \lim\limits_{n\to\infty} \frac{\left(\frac{n^3+20n^2+100}{5n^7-n-10}\right)}{\left(\frac{1}{n^4}\right)} =$

2

5

$$= \lim_{n\to\infty} \frac{n^7+20n^6+100n^4}{5n^7-n-10} = \lim_{n\to\infty} \frac{1+\frac{20}{n}+\frac{100}{n^3}}{5-\frac{1}{n^6}-\frac{10}{n^7}} = \frac{1}{5}.$$

Since $\lim\limits_{n\to\infty} \frac{a_n}{b_n}$ exists (is a finite real number) and $\sum b_n$

5 converges, then by the limit comparison test, $\sum a_n$ also $\boxed{\text{converges}}$.

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3. *(20 points)* Determine whether the series $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$ is absolutely convergent, conditionally convergent, or divergent. Give reasons for your answer.

Since $\frac{1}{n \ln n}$ is non-negative, decreasing, and has limit 0 as $n \to \infty$, Then $\sum \frac{(-1)^n}{n \ln n}$ converges by The Alternating Series Test. 6

For $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$, we can use The Integral Test.*

$$\int_2^{\infty} \frac{1}{x \ln x}\,dx \; (2) = \lim_{b \to \infty} (2) \int_2^{b} \frac{1}{x \ln x}\,dx = \lim_{b \to \infty} \int_{\ln 2}^{\ln b} \frac{1}{u}\,du \; (2) =$$

$$\boxed{\begin{array}{l} u = \ln x \\ du = \frac{1}{x}\,dx \end{array}} \; (2)$$

$$= \lim_{b \to \infty} \Big[\ln u\Big]_{u=\ln 2}^{u=\ln b} (2) = \lim_{b \to \infty} \left[\ln(\ln b) - \ln(\ln 2)\right] = \infty. \; (2)$$

So $\sum \frac{1}{n \ln n}$ diverges by the Integral Test.

Therefore $\sum \frac{(-1)^n}{n \ln n}$ $\boxed{\text{converges conditionally}}$. (2)

* (because $f(x) = \frac{1}{x \ln x}$ is non-negative and decreasing for $x \geq 2$)

4. *(10 points)* Using the formula for the sum of a geometric series, write down a power series expansion for the function $\frac{1}{1+x^5}$. For which values of x does the series converge?

We know that

(2) $\frac{1}{1-r} = 1 + r + r^2 + r^3 + \dots$, and the series converges for $|r| < 1$.

Putting $r = (-x^5)$, (3) we get

$$\frac{1}{1+x^5} = 1 + (-x^5) + (-x^5)^2 + (-x^5)^3 + \dots$$
$$= 1 - x^5 + x^{10} - x^{15} + \dots \quad (3)$$

which converges for $|x|^5 < 1$, or in other words (2) for $|x| < 1$.

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5. *(30 points)* For the power series

$$\sum_{n=1}^{\infty} \frac{5^n x^n}{n^2},$$

a. Find the radius of convergence.

[16]

Let $a_n = \frac{5^n x^n}{n^2}$. Then $L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{5^{n+1} x^{n+1}}{(n+1)^2} \cdot \frac{n^2}{5^n x^n} \right| =$ (2) (2) (2) (2)

$= \lim_{n\to\infty} \frac{5^{n+1}}{5^n} \frac{|x|^{n+1}}{|x|^n} \cdot \left(\frac{n}{n+1}\right)^2 = \lim_{n\to\infty} 5 \cdot |x| \cdot \left(\frac{1}{1+\frac{1}{n}}\right)^2 =$ (2)

$= 5|x| \cdot 1 = 5|x|$. (2) So by the ratio test, the series converges if $5|x| < 1$, or $|x| < \frac{1}{5}$; and diverges if $|x| > \frac{1}{5}$. (2)

The radius of convergence is therefore $\boxed{\frac{1}{5}}$. (2)

[12] b. Check each of the endpoints of the interval of convergence to see whether the series converges there. Explain your reasoning.

(2) The endpoints are at $x = \frac{1}{5}$ and $x = \frac{-1}{5}$.

* at $x = \frac{1}{5}$, the series is $\sum_{n=1}^{\infty} \frac{5^n \cdot (\frac{1}{5})^n}{n^2} = \sum \frac{1}{n^2}$, (2)

which converges (2) (it is a p-series with $p > 1$).

* at $x = -\frac{1}{5}$ the series is $\sum \frac{5^n (\frac{-1}{5})^n}{n^2} = \sum \frac{(-1)^n}{n^2}$. (2)

You could either use the Alt. Series test to say this (2) converges, or use Judy's Theorem (since $\sum \left|\frac{(-1)^n}{n^2}\right| = \sum \frac{1}{n^2}$ converges (2), then $\sum \frac{(-1)^n}{n^2}$ converges, because Absolute convergence implies convergence.)

c. What is the interval of convergence?

[2] From a), the series converges for $-\frac{1}{5} < x < \frac{1}{5}$ and diverges for $x < \frac{-1}{5}$ and for $x > \frac{1}{5}$.

From b), the series converges for $x = \frac{-1}{5}$ and $x = \frac{1}{5}$.

So the interval of convergence is $\boxed{-\frac{1}{5} \le x \le \frac{1}{5}}$.

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6. *(15 points)* Find the Taylor series for $f(x) = \sin x$ centered at the value $a = \pi$. Write out the first few terms of the series.

2

We have $f(x) = \sum_{n=0}^{\infty} c_n (x-\pi)^n$ where

$c_n = \dfrac{\frac{d^n f}{dx^n}(\pi)}{n!}$. 3 We compute c_n:

n	$\frac{d^n f}{dx^n}(x)$	$\frac{d^n f}{dx^n}(\pi)$	c_n	
0	$\sin x$	0	0	2
1	$\cos x$	-1	$-1/1 = -1$	2
2	$-\sin x$	0	0	2
3	$-\cos x$	$+1$	$+1/3!$	2
4	$\sin x$	0	0	2
5	$\cos x$	-1	$-1/5!$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	

The 2nd and 3rd columns repeat every 4 rows, so we see that $c_n = \dfrac{\pm 1}{n!}$ when n is odd, with signs alternating from one odd number to the next(*), and $c_n = 0$ when n is even.

We have

$$\sin x = 0 + (-1)(x-\pi) + 0 + \frac{1}{3!}(x-\pi)^3 + 0 + \frac{-1}{5!}(x-\pi)^5 + \dots$$

$$= \boxed{-(x-\pi) + \frac{(x-\pi)^3}{3!} - \frac{(x-\pi)^5}{5!} + \frac{(x-\pi)^7}{7!} - \dots}$$ 5

(*) A shorter way to say this is: $c_{2k+1} = \dfrac{(-1)^{k-1}}{(2k+1)!}$ for $k = 0, 1, 2, 3, \dots$