EXAM 2
Math 2924
3-27-15

Name *key*

Instructions *Work all of the following problems in the space provided. If there is not enough room, you may write on the back sides of the pages. Give thorough explanations to receive full credit.*

1. Find the indefinite integrals:

a. *(15 points)* $\int x^6 \ln x \, dx$

$$\boxed{\begin{array}{ll} u = \ln x & dv = x^6\,dx \\ du = \frac{1}{x}\,dx & v = \frac{x^7}{7} \end{array}}$$

5

4

$$= \int u\,dv = uv - \int v\,du$$

2

$$= (\ln x)\cdot\frac{x^7}{7} - \int \frac{x^7}{7}\cdot\frac{1}{x}\,dx$$

2

$$= \frac{x^7 \ln x}{7} - \frac{1}{7}\int x^6\,dx$$

2

$$= \frac{x^7 \ln x}{7} - \frac{1}{7}\cdot\frac{x^7}{7} + C$$

$$= \boxed{\frac{x^7 \ln x}{7} - \frac{x^7}{49} + C}$$

b. *(20 points)* $\int \frac{1}{x^2\sqrt{4-x^2}}\,dx$

(You may want to use the formula $\int \csc^2\theta = -\cot\theta + C$)

$x = 2\sin\theta$ 2

$dx = 2\cos\theta\,d\theta$ 2

1 2

$$\int \frac{1}{x^2\sqrt{4-x^2}}\,dx = \int \frac{1}{4\sin^2\theta\sqrt{4-4\cos^2\theta}}\cdot 2\cos\theta\,d\theta$$

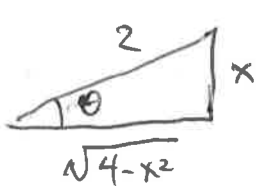
2

$$= \frac{2}{4\sqrt{4}}\int \frac{\cos\theta}{\sin^2\theta\sqrt{1-\cos^2\theta}}\,d\theta = \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta\cos\theta}\,d\theta$$

2 2 2

$$= \frac{1}{4}\int \frac{d\theta}{\sin^2\theta} = \frac{1}{4}\int \csc^2\theta\,d\theta = \frac{1}{4}(-\cot\theta + C)$$

$$= \boxed{\frac{1}{4}\left(\frac{-\sqrt{4-x^2}}{x} + C\right)}$$



3

2

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c. *(20 points)* $\int \frac{1}{x^2(x+1)}\,dx$

Put $\frac{1}{x^2(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1}$ 4

Then $1 = \frac{A(x^2)(x+1)}{x} + \frac{Bx^2(x+1)}{x^2} + \frac{Cx^2(x+1)}{x+1}$,

or $1 = Ax(x+1) + B(x+1) + Cx^2$ 3

Put $x=-1$: $1 = 0 + 0 + C\cdot 1 \Rightarrow C=1$ 2

$x=0$: $1 = 0 + B\cdot 1 + C \Rightarrow B = 1$ 2

$x=1$: $1 = A\cdot 1\cdot 2 + B\cdot 2 + C \Rightarrow 1 = 2A + 2B + C$

$\Rightarrow 1 = 2A + 2 + 1 \Rightarrow 2A = -2$

$\Rightarrow A = -1$. 3

So $\frac{1}{x^2(x+1)} = \frac{-1}{x} + \frac{1}{x^2} + \frac{1}{x+1}$

and $\int \frac{1}{x^2(x+1)}dx = -\int \frac{dx}{x} + \int \frac{dx}{x^2} + \int \frac{dx}{x+1} = \boxed{-\ln|x| + \frac{x^{-1}}{-1} + \ln|x+1| + C}$

2 2 2

2. *(15 points)* Evaluate the improper integral $\int_1^\infty e^{-2x}\,dx$, showing all work.

$\int_1^\infty e^{-2x}dx = \lim_{b\to\infty} \int_1^b e^{-2x}dx = \lim_{b\to\infty} \int_{-2}^{-2b} e^{+u}\frac{du}{-2} = \lim_{b\to\infty}\left(\frac{-1}{2}\int_{-2}^{-2b} e^u du\right)$ 2 2

2 $\boxed{\begin{matrix} u=-2x \\ du = -2dx \end{matrix}}$ $\boxed{\begin{matrix} x=1 \rightarrow u=-2 \\ x=b \rightarrow u=-2b \end{matrix}}$ 2

$= \lim_{b\to\infty} \frac{-1}{2}\left[e^u\right]_{-2}^{-2b} = \lim_{b\to\infty} \frac{-1}{2}\left(e^{-2b} - e^{-2}\right)$ 2 2

$= \frac{-1}{2}\left(0 - e^{-2}\right) = \frac{+e^{-2}}{2} = \boxed{\frac{1}{2e^2}}$

3

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3. *(15 points)* Use the comparison theorem to determine whether the improper integral $\int_1^\infty \frac{1}{1+x^5}\,dx$ is convergent or divergent. Give a complete explanation with your answer.

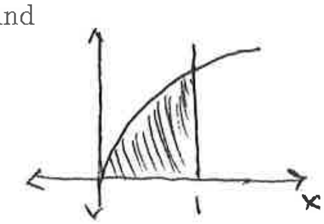
We have $1+x^5 \ge x^5$, so $\frac{1}{1+x^5} \le \frac{1}{x^5}$ for all $x \ge 1$. 4

Also $\int_1^\infty \frac{1}{\cancel{x^5}}\,dx = \lim_{b\to\infty} \int_1^b \frac{1}{x^5}\,dx = \lim_{b\to\infty} \int_1^b x^{-5}\,dx =$ 2

$= \lim_{b\to\infty} \left[\frac{x^{-4}}{-4}\right]_1^b = \lim_{b\to\infty}\left[\frac{b^{-4}}{-4} - \frac{1}{-4}\right] = \lim_{b\to\infty}\left[\frac{-1}{4b^4} + \frac{1}{4}\right] =$ 2

$= 0 + \frac{1}{4}$, a finite limit 4. So $\int_1^\infty \frac{1}{x^5}\,dx$ converges, and therefore by the comparison test $\int_1^\infty \frac{1}{1+x^5}\,dx$ converges also. 3

4. *(15 points)* The region shaded in the diagram is between the graph of $y = \sqrt{x}$, the x-axis, and the line $x = 1$. Assume the region is occupied by a solid with density $\rho = 1$. For this solid, find



a. the mass M (use $M = \int y\,dx$)

$$M = \int_0^1 \sqrt{x}\,dx = \int_0^1 x^{1/2}\,dx = \left[\frac{2x^{3/2}}{3}\right]_0^1 = \boxed{\frac{2}{3}}.$$

2 2 1

b. the moment about the y-axis (use $M_y = \int xy\,dx$)

$$M_y = \int_0^1 x\sqrt{x}\,dx = \int_0^1 x^{3/2}\,dx = \left[\frac{2}{5}x^{5/2}\right]_0^1 = \boxed{\frac{2}{5}}$$

2 2 2 1

c. the x-coordinate $\bar{x}$ of the center of mass (use $\bar{x} = M_y/M$)

$$\bar{x} = \frac{2/5}{2/3} = \boxed{\frac{3}{5}}.$$

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