EXAM 1 Name Key

Math ~~2433~~ 2924

~~9-22-14~~ 2-13-15

Instructions *Work all of the following problems in the space provided. If there is not enough room, you may write on the back sides of the pages. Give thorough explanations to receive full credit.*

1. *(8 points)* Prove that $\frac{d}{dx}e^x = e^x$. You may use the fact that $\lim_{h\to 0} \frac{e^h - 1}{h} = 1$.

[8]

$$\frac{d}{dx}(e^x) = \lim_{h\to 0}\left(\frac{e^{x+h} - e^x}{h}\right) = \lim_{h\to 0}\left(\frac{e^x e^h - e^x}{h}\right)$$

$$= \lim_{h\to 0} e^x\left(\frac{e^h - 1}{h}\right) = e^x \lim_{h\to 0}\left(\frac{e^h-1}{h}\right) = e^x \cdot 1 = e^x.$$

(1) (1) (1) (1) (1) (1) (1) (1)

2. *(8 points)* Prove that $\frac{d}{dx}(\arctan x) = \frac{1}{x^2+1}$.

[8]

Let $y = \arctan x$. Then $x = \tan y$, so $\frac{d}{dx}(x) = \frac{d}{dx}(\tan y)$,

so $1 = \sec^2 y \frac{dy}{dx}$, so $\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1+\tan^2 y} = \frac{1}{1+x^2}$.

(2) (2) (2) (1) (1)

3. *(16 points)* Find the derivative. Give the answer as a function of x.

a. $y = \ln(e^x \arctan x)$

[6]

$$\frac{dy}{dx} = \frac{1}{e^x \arctan x} \cdot \frac{d}{dx}(e^x \arctan x) \quad (2)$$

$$\Rightarrow \frac{dy}{dx} = \boxed{\frac{1}{e^x \arctan x} \cdot \left(e^x \arctan x + e^x \cdot \frac{1}{1+x^2}\right)}$$

(2) (2)

b. $y = (\tan x)^x$

[10]

$\ln y = x\ln(\tan x)$, so $\frac{d}{dx}(\ln y) = \frac{d}{dx}(x\ln(\tan x))$, (2)

$$\frac{1}{y}\cdot\frac{dy}{dx} = 1\ln(\tan x) + \frac{x}{\tan x}\cdot \sec^2 x \quad (2)\ (2)$$

$$\frac{dy}{dx} = y\left(\ln(\tan x) + \frac{x\sec^2 x}{\tan x}\right) \quad (2)$$

$$\frac{dy}{dx} = (\tan x)^x\left(\ln(\tan x) + \frac{x\sec^2 x}{\tan x}\right) \quad (1)$$

 Name ______________

4. *(16 points)* For the function $f(x) = x^7 \ln x$:

a. Find the critical point of the function.

[10] $f'(x) = 7x^6 \ln x + x^7 \cdot \frac{1}{x} = 7x^6 \ln x + x^6 = x^6(7\ln x + 1)$ (2)

Critical point is where $f'(x) = 0$, and $x \neq 0$ in the domain of f,

so $(7\ln x + 1) = 0$, so $\ln x = \frac{-1}{7}$, or $\boxed{x = e^{-1/7}}$.

(2) (2) (2)

b. Find the value of $f''(x)$ at the critical point. Simplify as much as possible.

[6] $f''(x) = 6x^5(7\ln x + 1) + x^6\left(7 \cdot \frac{1}{x}\right)$ (2)

or $f''(x) = 42x^5 \ln x + 6x^5 + 7x^5 = 42x^5 \ln x + 13x^5$

or $f''(x) = x^5(42\ln x + 13)$. (2) When $x = e^{-1/7}$, then

$f''(e^{-1/7}) = (e^{-1/7})^5\left(42 \cdot \left(\frac{-1}{7}\right) + 13\right) = e^{-5/7}(-6+13) = \boxed{e^{-5/7} \cdot 7}$

(2)

(2 points extra credit): Is the critical point a maximum or minimum? Justify your answer.

[+2] Since $f'' > 0$ at the critical point, it is a minimum by the second derivative test.

5. *(16 points)* Find the indefinite integrals:

[8] **a.** $\int \frac{e^x}{\sqrt{e^x + 1}}\, dx$ $= \int \frac{du}{\sqrt{u}} = \int u^{-1/2}\, du = \frac{u^{+1/2}}{\frac{1}{2}} + C$ (2)

(1)

$\boxed{\text{Let } u = e^x + 1,\ du = e^x\, dx}$ (4)

$= 2u^{\frac{1}{2}} + C$ (1)

$= \boxed{2\sqrt{e^x + 1} + C}$

(1)

[8] **b.** $\int \frac{1}{x} \frac{1}{\sqrt{1 - (\ln x)^2}}\, dx$ $= \int \frac{1}{\sqrt{1-u^2}}\, du = \arcsin u + C$ (2)

$= \boxed{\arcsin(\ln x) + C}$ (1)

$\boxed{\text{Let } u = \ln x,\ du = \frac{1}{x}\, dx}$ (4)

 Name ____________________

6. *(16 points)* Evaluate the limit, showing all work.

[8] **a.** $\lim_{x\to 0} \frac{e^{3x}-1}{\sin x}$ $= \lim_{x\to 0} \frac{\frac{d}{dx}(e^{3x}-1)}{\frac{d}{dx}(\sin x)} = \lim_{x\to 0} \frac{3e^{3x}}{\cos x} = \frac{3e^0}{\cos 0}$ 2 2

(of type $\frac{e^0-1}{0} = \frac{1-1}{0} = \frac{0}{0}$, so L'hopital's rule applies) 2

$= \frac{3\cdot 1}{1} = \boxed{3}$ 2

[8] **b.** $\lim_{x\to\infty} \frac{\ln x}{x^2}$ $= \lim_{x\to\infty} \frac{\frac{d}{dx}(\ln x)}{\frac{d}{dx}(x^2)} = \lim_{x\to\infty} \frac{\left(\frac{1}{x}\right)}{2x} = \lim_{x\to\infty} \frac{1}{2x^2} = \frac{1}{\infty} = \boxed{0}$ 2 2 2 2

(of type $\frac{\infty}{\infty}$)

7. *(20 points)* Find the definite integrals. Simplify your answer as much as possible.

[10] **a.** $\int_0^{\pi/2} \frac{\sin x}{\cos x + 7}\,dx$ $= \int_8^7 \frac{\sin x}{u} \frac{du}{(-\sin x)} = -\int_8^7 \frac{du}{u}$ 2

$u = \cos x + 7$ 1
$du = -\sin x\,dx$ 1
$x = 0 \to u = \cos 0 + 7 = 8$ 1
$x = \frac{\pi}{2} \to u = \cos\frac{\pi}{2} + 7 = 7$ 1

$= -\left[\ln u\right]_8^7$ 2

$= -\left[\ln 7 - \ln 8\right] = \ln 8 - \ln 7$

2 $= \boxed{\ln(8/7)}$

[10] **b.** $\int_{\pi/2}^{\pi} \frac{\sin x}{\cos^2 x + 1}\,dx$ $= \int_0^{-1} \frac{\sin x}{u^2+1} \frac{du}{-\sin x}$

$u = \cos x$ 1
$du = -\sin x\,dx$ 1
$x = \frac{\pi}{2} \to u = \cos\frac{\pi}{2} = 0$ 1
$x = \pi \to u = \cos\pi = -1$ 1

$= -\int_0^{-1} \frac{du}{u^2+1} = \int_{-1}^{0} \frac{du}{u^2+1}$ 2

$= \left[\arctan u\right]_{-1}^{0}$ 2

$= \arctan 0 - \arctan(-1)$

$= 0 - \left(\frac{-\pi}{4}\right) = \boxed{\frac{\pi}{4}}$ 2